MA 406
General topology

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Tuesday
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grading: 4 Quizes - 40%
midsem - 20%
endsem - 40%

Quiz - 22^{nd} Jan
1st week Feb
end of feb
and one more

100 - AP, 90 ≥ AA, 80 ≥ AB,
70 ≥ BB, 60 ≥ BC....

Attendence - Not comp, but necc

<u>Textbook</u>: Munkres
→ HW / Quiz / midsem / endsem

<u>Re-exam</u>: No reexam unless medical reasons

lecture notes - on website + HW

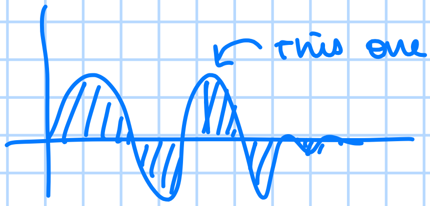
7th Jan :

Topology : study of continuous function on a set X.

This is quiet different from calculus.

eg :

Suppose if $f:[0,1]\to\mathbb{R}$, what is $\int_0^1 f(t)\,dt$



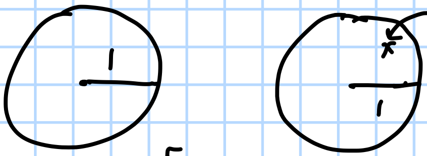
This one point does not make any difference

so $\int_0^1 f = \int_{[0,1]\setminus\{t\}} f$ ← some point

Note : Topological point of view, the following are fundamentally different

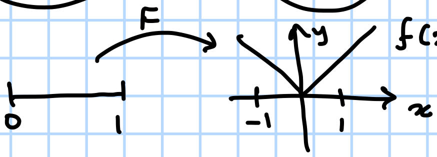
① $[0,1]$, $[0,\tfrac{1}{2})\cup(\tfrac{1}{2},1]$

②



Punctured disk

③



$f(x)=|x|$

← They are the same things (homeomorphism)

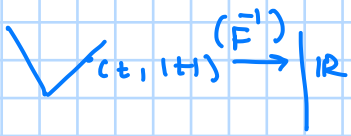
intuitively we are allowed to strech/Bend but not Break

problem : let $X=\{(t,|t|)\,;\,-1<t<1\}$

Is there a map $F:[0,1]\to X$ s.t F is

done ① Bijective

② F and F^{-1} is differential



Assume $X\neq\phi$ → Ask

Def : A topology on a set X is a collection of subsets of X, denote it by τ, so that it is closed under a finite intersection, if $A_1, A_2 \in \tau$ then

①

$A_1\cap A_2\ldots\cap A_n\in\tau$

② Closed under arbitrary union , if $A_\alpha\in\tau$

$\bigcup A_\alpha\in\tau$

③ $\phi, X\in\tau$

Note : elements of τ are called open sets

eg : ① X take $\tau=2^X$ ← power set of X

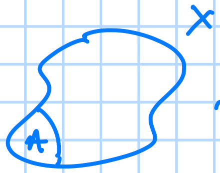
② $\tau=\{\phi, X\}$

③ $X=\{0\}$ ← set with 1 element , then trivial

④ suppose $X=\mathbb{R}$

$\tau=\{\{1\},\phi,\mathbb{R}\}$

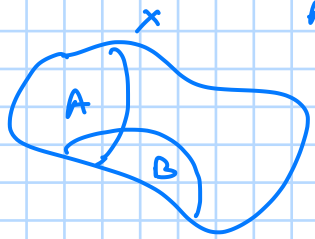
⑤ Take any $|x|>1$, let



then ↙ A is non-empty

$\tau=\{A,\phi,X\}$

⑥ let $|X| > 2$, let $A, B \subseteq X$



A, B are open
or
$A, B \in \tau$
then
$A \cup B \in \tau$
$A \cap B \in \tau$
$\Rightarrow$

HW: check τ is a topology $\tau = \{A, B, A\cup B, A\cap B, \phi, X\}$

$\rightarrow$ Done

Now, given $A_1, A_2 \dots A_{10} \subseteq X$

let $A_{11} = \phi$
$A_{12} = X$

then

$$\tau_1 = \{B \mid B = A_{\alpha_1} \cap \dots A_{\alpha_m} \text{ for some } m > 1\}$$ ← All possible intersections

now consider $\tau = \{C \mid C = \bigcup_{\alpha \in I} B_\alpha, \; B_\alpha \in \tau_1\}$

Claim: The above defined τ is a topology on X

proof: ← ask ← How to write it

Eg: ⑦ given $\mathcal{A}$ collection of subsets of X. let

Note: if we don't take finite, we may get more sets

$\tau_1 = \{$All posible finite intersection members of $\mathcal{A}$, ϕ, $X\}$

$\tau = \{$All posible union of elements of $T_1\}$

Eg: ⑧ $X = B(0,1)$, $\mathcal{A} = \{B(0, \frac{1}{n}), n \geq 1\} \cup \{\phi\}$

now, $\tau_1 = \mathcal{A} \cup \{0\}$ ← singleton set becomes an open set in $\mathbb{R}^2$

↑ All possible intersections

this is why we take finite:

$\tau_1 = \mathcal{A}$

← for finite intersections as $\{0\}$ does not make sense

Ex: ⑨ $X = \mathbb{R}$
τ = usual topology i.e

A is open if $A = \bigsqcup_{i \in I} (\alpha_i, \beta_i)$ ← disjoint

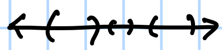
← open set

claim: Any open set A in $\mathbb{R}$ is a countable union of disjoint open intervals

proof:

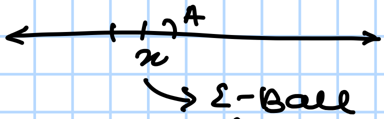
A is open in $\mathbb{R}$, there

← union of all



or A if open can disjoint into countable union of disjoint open sets.

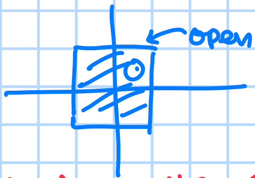
A is open in $\mathbb{R}$ if given $\forall x \in A, \exists \varepsilon = \varepsilon(x) > 0$
s.t $(x - \varepsilon, x + \varepsilon) \subseteq A$



given $x \in A$, let I_x be the largest interval s.t $x \in I_x \subseteq A$ — It contains rationals so countable

HW: Prove that openset indeeds gives us topology on $\mathbb{R}$ → done

Q: what happen to claim 1 in $\mathbb{R}^2$



open set → done

HW: is it possible to write $(-1,1)\times(-1,1) = \bigsqcup_{i \in I} B(x_i, \varepsilon_i)$

Eg: (9) $X = \mathbb{R}^2$

$$\tau = \{A \mid A = \bigcup B(x,r)\} \cup \{\phi, \mathbb{R}^2\}$$

$$B(x,r) = \{ y = (y_1, y_2) \mid \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} < r \}$$

claim: τ mentioned above is a topology

proof: By constrution

① $\phi, \mathbb{R}^2 \in \tau$

② finite intersection: want to prove that given $A_1, \ldots, A_n \in \tau$, $A_1 \cap \cdots \cap A_N \in \tau$

By induction lets first prove on $N=2$

$$A_1 = \bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$$

$$A_2 = \bigcup_{\beta \in J} B(x_\beta, r_\beta)$$

now, $A_1 \cap A_2 = \left[\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)\right] \cap \left[\bigcup_{\beta \in J} (x_\beta, r_\beta)\right]$

$$= \bigcup_{\alpha \in I} \bigcup_{\beta \in J} \left[B(x_\alpha, r_\alpha) \cap B(y_\beta, r_\beta) \right]$$

It is sufficient to prove that $B_1 \cap B_2$ can be written as union of Balls.

$B(P, \frac{1}{100}) \subseteq B_2$

$B(P, \frac{1}{1000}) \subseteq B_2$ → we choose the smaller one

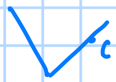
Ask about infinity

problem: let $X = \{(t, |t|) ; -1 < t < 1\}$

Is there a map $F : [0,1] \to X$ s.t F is

① Bijective

② F and F^{-1} is differential



$(t, |t|) \xrightarrow{(F^{-1})} \mathbb{R}$

Ans: $F : [0,1] \to X$

$$X = \{(t, |t|) \mid -1 < t < 1\}$$

now if $[0,1] \xrightarrow{\times 2, -1} [-1, 1]$

or

$2[0,1] - [0,1]$

variable

$x \in [0,1]$

$F(x) = 2x - 1$

$2x-1, |2x-1|$

$$F(x) = (2x-1, |2x-1|)$$

$$F^{-1}(x, |x|) = \frac{x+1}{2}$$

$$FF^{-1}(x,|x|) = F\left(\frac{x+1}{2}\right) = \left(2\left(\frac{x+1}{2}\right) - 1, |x|\right)$$

$$= (x, |x|)$$

$$F^{-1}F(x) = F^{-1}(2x-1, |2x-1|) = \cancel{2}\left(\frac{x+1}{\cancel{2}}\right) - 1 = x$$

$$F^{-1}(x,|x|) = \frac{x+1}{2}$$

HW: check τ is a topology $\tau = \{A, B, A\cap B, A\cup B, \phi, X\}$

Ans: As ① $\phi, X \in \tau$

② for $A, B \in \tau$

$A \cap B \in \tau$

$A, A\cup B \in \tau$ same for B, $A \cup B$

$A \cap B \in \tau$

$A, A \cap B \in \tau$ same for B, $A \cap B$

till

$A \in \tau$

and same for three

③ same for union.

HW: Prove that openset indeeds gives us topology on $\mathbb{R}$

open sets gives us topology in $\mathbb{R}$ as

① $\phi, \mathbb{R}$ is both open and closed so $\phi, \mathbb{R} \in \tau$

② if A_i is open and say $A_1, A_2 \ldots A_n \in \tau$ then

as $\forall x_i \in A_i$, $\exists \delta > 0$ s.t $B_\delta(x_i) \subseteq A_i$

now for $A_1 \cap A_2 \ldots \cap A_n$
as $\forall x \in A_1 \cap \ldots \cap A_n$
$\Rightarrow x \in A_i \ \forall i$
$\Rightarrow \exists \delta_i$ s.t $B_{\delta_i}(x) \subseteq A_i$

then $\delta = \min\{\delta_i\}$ is s.t
$B_\delta(x) \subseteq A_i \ \forall i$

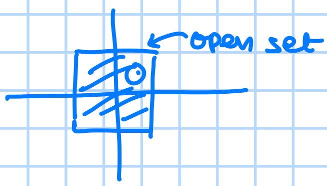
$\Rightarrow B_\delta(x) \subseteq \cap A_i$
$\Rightarrow \cap A_i$ is open

$\Rightarrow \cap A_i \in \tau$

③ union of infinite many open set is open as any point in the union is in atleast one open set so $\exists\ \delta$ s.t the δ Ball is in the union

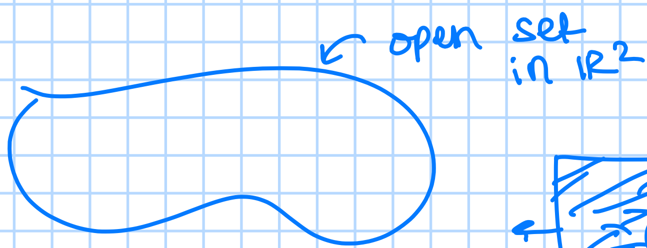
so arbitrary union of sets in $\tau \in \tau$

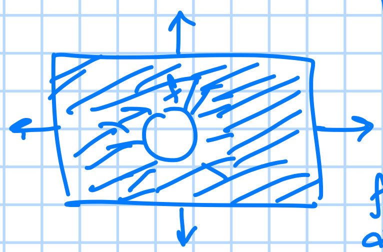
Q: what happens to claim 1 in $\mathbb{R}^2$



To show that if open set in $\mathbb{R}^2$ is countable union of disjoint open sets

as $\bar{B}(0,1)$ is a closed set
$\mathbb{R}^2 \setminus \bar{B}(0,1)$ is open.

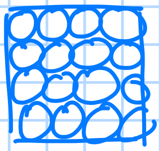




if $\mathbb{R}^2 \setminus \bar{B}(0,1)$ can be written as countable union of disjoint sets then for any one of these disjoint set, one of the point on the boundary of this point is not in the set itself but in some other set which is not possible as all sets are disjoint.

HW: is it possible to write $(-1,1)\times(-1,1) = \bigsqcup_{i\in I} B(x_i,\varepsilon_i)$

Ans: $(-1,1)\times(-1,1) = \bigsqcup_{i\in I} B(x_i,\varepsilon_i)$



if this is possible then one of the balls will have $(1,1)$ as boundary but if $(1,1)$ is a boundary point and as it is a corner, this is not possible so this is a contradiction.

10th Jan :

Ex: ① A metric space

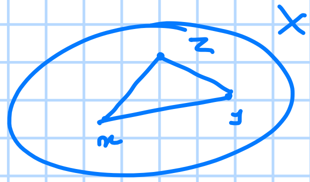
(X, d) s.t $d : X \times X \longrightarrow [0, \infty)$

① $d(x, y) \geq 0$

② $d(x, y) = 0 \iff x = y$

③ $d(x, y) = d(y, x)$

④ $d(x, y) \leq d(x, z) + d(z, y)$



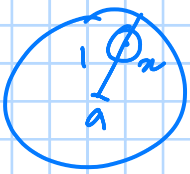
In X, $a \in X$ $\quad B(a, r) = \{ x \in X \mid d(a, x) < r \}$

Recall: $A \subseteq X$ was open if $\forall a \in A$, $\exists r = r(a) > 0$ s.t $B(a, r) \subseteq A$

Theorm 1: let (X, d) be a metric space, then $\mathcal{B} = \{ A \subseteq X \mid A \text{ is open} \}$ is a topology on X

claim: suppose $a \in X$ then $B(a, 1)$ is an open set.

proof:



let $x \in B(a, 1)$. Want an $r > 0$ s.t $B(x, r) \subseteq B(a, 1)$

now $d(x-a) < 1$ as $x \in B(a, 1)$

let $d(a, x) = 1 - \varepsilon$ then

$B(x, \varepsilon/2) \subseteq B(a, 1)$ by triagle inequality

this means in metric space open balls are open

proof: For Theorm 1, as $B(a, 1)$ is open set all balls are in $\mathcal{B}$ (all open ball are open)

① also by defination $\phi, X \in \mathcal{B}$

② $\bigcup_{m,n}$: Trivial (union of open balls is open)

③ Finite intersection

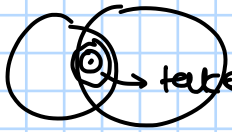
Enough to prove that if A, B are open then so is $A \cap B$ is open

$A = \bigcup_{a \in A} B(a, r_a) \leftarrow$ By defination (union of open balls by defn)

$B = \bigcup_{b \in B} B(b, r_b) \leftarrow$ By defination (union of open balls by defn)

now, $\quad A \cap B = \bigcup_{a \in A} \bigcup_{b \in B} \underbrace{[B(a, r_a) \cap B(b, r_b)]}_{\text{this is intersection of balls}}$

proved before



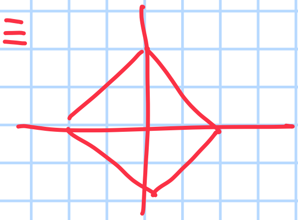
take the smaller one

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$ all are topological spaces.

Note: On $\mathbb{R}^2$ $\|(x_1, y_1) - (x_2, y_2)\| =$ euclidian distance

l_1 norm $d((x_1, y_1) - (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$

$B((0,0), 1) \equiv$



Is the topological space same

let τ be topology on $(\mathbb{R}^2, \|\cdot\|_1)$

Que: Is $(\mathbb{R}^2, \tau) = (\mathbb{R}^2, \text{standard topology})$

Ans: Yes. Any set open in ($\mathbb{R}^2$, standard topology) is also open in $(\mathbb{R}^2, \| \ \|_1)$ and vica versa.

Claim: $\exists\, c_1, c_2 > 0$ s.t for $p, q \in \mathbb{R}^2$ ← two norms are equivalent

$$c_2 \|p - q\| \leq \|p - q\|_1 \leq c_1 \|p - q\|$$

proof:

$|x_1 - x_2| + |y_1 - y_2| \leq \sqrt{2}\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)$ ← By AP $\geq$ GP

special case and $\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$

$\leq (|x_1 - x_2| + |y_1 - y_2|)^2$

$\Rightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq |x_1 - x_2| + |y_1 - y_2|$ $\quad c_1 = \sqrt{2},\ c_2 = 1$

Theorm: If $\|\cdot\|_1$ and $\|\cdot\|_2$ on V, where $\dim V < \infty$. Then

$\exists\, c_1, c_2 > 0$ s.t

$$c_1 \|x\|_2 \leq \|x\|_1 \leq c_2 \|x\|_2$$

proof:

Here norm $\| \ \| : V \to [0, \infty)$

① $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$
② $\|x + y\| \leq \|x\| + \|y\|$
③ $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in \mathbb{R}$

norms:

$\sqrt{x^2 + y^2}$, $\underbrace{|x| + |y|}_{l_1}$, $\underbrace{(|x|^p + |y|^p)^{\frac{1}{p}}}_{l_p}$, $\underbrace{\max(|x|, |y|)}_{\text{sup norm}}$

special case $V = \mathbb{R}^2$

lets prove

$$c_2 \sqrt{x^2 + y^2} \leq \|x, y\| \leq c_1 \sqrt{x^2 + y^2}$$

$\|x, y\| = \|x e_1 + y e_2\| \leq |x| \|e_1\| + |y| \|e_2\|$

$(|x| \|e_1\| + |y| \|e_2\|)^2 \leq (|x|^2 + |y|^2)(\|e_1\|^2 + \|e_2\|^2)$ (cauchy shwarts)

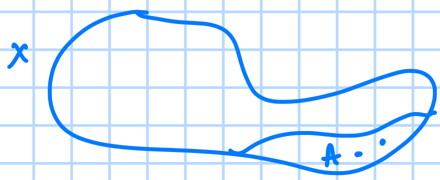
so $\|x, y\| \leq \sqrt{x^2 + y^2} \ \underbrace{\sqrt{\|e_1\|^2 + \|e_2\|^2}}_{c_1}$

we want to prove that
if $x^2 + y^2 = 1$ then
$\|(x,y)\| \geq c_2$
then
look at
$$\left\| \frac{(x,y)}{\sqrt{x^2+y^2}} \right\| \geq c_2$$

$$\big| \|(x,y)\| - \|(z,w)\| \big| \leq \|(x,y) - (z,w)\|$$

take $c_2 = \min \| \ \|$
so we get $c_1 \leftarrow$ triangle ineq
and then unit sphere is compact
so $\min \| \ \| = c_2 \rightarrow$ exist

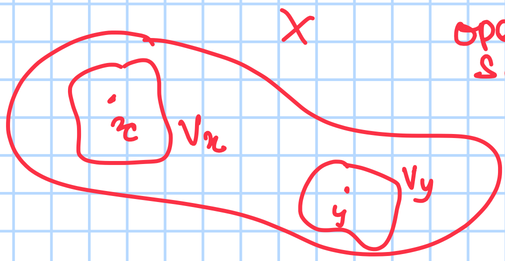
Exe: Are there (X, τ) which is not a metric space
Ans: Yes.



$\tau = \{\phi, A, X\}$ here X does not have to be metric space

Def: A space (X, τ) is called Hausdorff if $x \neq y$, $\exists$ open set V_x, V_y so that $x \in V_x$, $y \in V_y$,

$V_x \cap V_y = \phi$



Exe: Is there (X, τ) which is Hausdorff but is not a metric space?
Ans: Trivial τ
Exe: How do we know if given (X, τ) is metric space or not?

Defn: A collection of open set $\mathcal{B}$ is called basis of (X, τ) if

① $X = \bigcup_{B \in \mathcal{B}} B$

② $\mathcal{B}$ is almost closed under intersection

given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ $\exists$ $B_3 \in \mathcal{B}$
s.t $x \in B_3 \subseteq B_1 \cap B_2$

Exe: ① $\mathcal{B} = \{ B(a, r) \mid a \in \mathbb{R}^2, r > 0 \}$

Theorm: let $(X, \tau_1), (X, \tau_2)$ be two topologies. If $\exists \mathcal{B}$ which is a basis for τ_1, τ_2 then $\tau_1 = \tau_2$

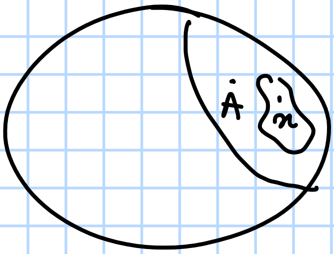
claim: Suppose A is open (X,τ) and $\mathcal{B}$ is basis for τ then

$$A = \bigcup_{\alpha \in I} B_\alpha$$

when each $B_\alpha \in \mathcal{B}$

so $\forall x \in A$, $\exists B_\alpha \in \mathcal{B}$ s.t $x \in B_x \subseteq A$

proof:



since $X = \bigcup_{B \in \mathcal{B}} B$

$$A = A \cap X = \bigcup_{B \in \mathcal{B}} (A \cap B)$$

now $x \in A \cap B$ for $B \in \mathcal{B}$

potential problem:

$$\mathcal{B} = \{ B(a,r) \mid a \in \mathbb{R}^2, r > 0 \}$$

14th Jan :

Senario-1 : suppose $X \neq \phi$, $\mathcal{A} \subseteq 2^X := \{A \subseteq X\}$, (Assume $\bigcup_{A \in \mathcal{A}} A = X$) $\exists$ initial topology τ on X which includes each $A \in \mathcal{A}$ as open-set

step 1 : $\mathcal{B} = \left\{ \bigcap_{i=1}^{N} A_i : A_i \in \mathcal{A}, N \geq 1 \right\}$ ← Think of this like molecules

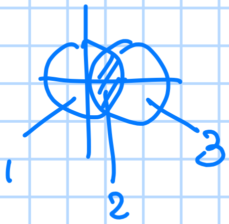
step 2 : $\tau = \left\{ \bigcup_{\alpha \in I} B_\alpha \;\middle|\; B_\alpha \in \mathcal{B}, I \text{ is an set} \right\}$

Eg : $X = \mathbb{R}^2$

$\mathcal{A} = \{ B(0,1),\ B((0,1),1),\ \mathbb{R}^2 \}$ $\quad n=3$

$\mathcal{B} = \left\{ \bigcap_{i=1}^{N} A_i \;\middle|\; A_i \in \mathcal{A}, N \geq 1 \right\} = \{ ①, ②, ③, \mathbb{R}^2 \}$

$N = 1, 2, 3$



$\tau = \{ A_1, A_2, A_1 \cap A_2, A_1 \cup A_2, \phi, \mathbb{R}^2 \}$

as I can be empty

Defn : (Basis) Given (X, τ), A collection of open sets is called a basis if each, $A \in \tau$ can be written

$$A = \bigcup_{\alpha \in I} B_\alpha$$

Eg : ($\mathbb{R}^2$, std topology)

natural basis: ① $\mathcal{B} = \{ B(p, r) \mid p \in \mathbb{R}^2, r > 0 \}$

also ② τ is itself a base

③ $\mathcal{B}$ in Senario-1 is also a basis.

Senario-2 :

suppose $\mathcal{B}' \subseteq 2^X$ where $X \neq \phi$, has a property that $X = \bigcup_{B \in \mathcal{B}'} B$ ← Basis construction

and it is "almost closed" under intersection, i.e given $B_1, B_2 \in \mathcal{B}'$

$$B_1 \cap B_2 = \bigcup_{B_\alpha \in \mathcal{B}'} B_\alpha$$

$\exists$ natural topology τ on X which makes each member of $\mathcal{B}'$ an open set given by ← Topology generated by $\mathcal{B}'$

$$\tau = \left\{ \bigcup_{\alpha \in I} B_\alpha \;\middle|\; B_\alpha \in \mathcal{B}' \right\}$$

claim : τ is a topology on X, where τ is as above.

proof: suppose $A_1, A_2 \in \tau$, we want to prove that $A_1 \cap A_2 \in \tau$

$$A_1 = \bigcup_{i \in I} B_i, \quad A_2 = \bigcup_{j \in J} C_j$$

$$B_i, C_j \in \mathcal{B}$$

$$A_1 \cap A_2 = \bigcup_{i,j} [B_i \cap C_j]$$

$$B_i \cap C_j = \bigcup_{k \in J_{i,k}} B_k$$

$$= \bigcup_{i,j} \Big[\bigcup_{k \in J_{i,k}} B_k \Big]$$

Note: Natural basis in Senario-2 basis is $\mathcal{B}'$

why Basis: There are lot of open sets in (X, τ), to study τ we need basis

Theorm: (2) Suppose τ, τ' are topologies on X. Then $\tau = \tau'$ if they share a common basis.

proof: let $\mathcal{B}$ be the common basis for τ, τ'

we have to show that if $A \in \tau$ then $A \in \tau'$

$$A = \bigcup_{\alpha \in I} B_\alpha, \quad B_\alpha \in \tau$$

$\in \tau$

$$\Rightarrow A = \bigcup_{\alpha \in I} B_\alpha, \quad B_\alpha \in \tau'$$

so $A \in \tau'$

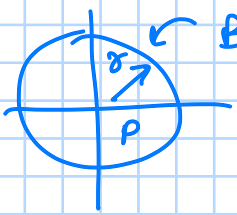
Theorm: (Application of theorm (2)) let $X = \mathbb{R}^2$ let τ_1 be the standard topology on $\mathbb{R}^2$, and τ_2 be the topology on $\mathbb{R}^2$ given by the metric

$$d((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|$$

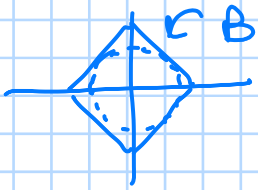
then $\tau_1 = \tau_2$

proof: $\mathcal{B} = \{ B(P, r) \mid P \in \mathbb{R}^2, r > 0 \}$ is a basis for $(\mathbb{R}^2, d)$ then we are done.

Ball in $\|\cdot\|$



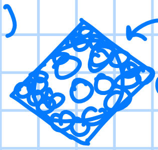
Ball in d



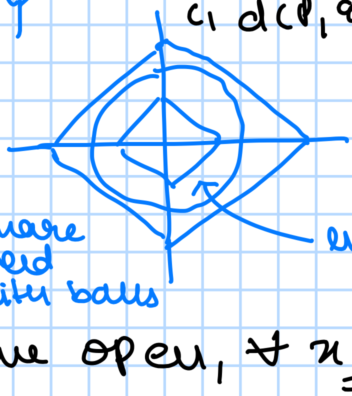
here recall that

$$c_1 d(P, q) \le \|P - q\| \le c_1 d(P, q)$$

every set in Basis of squas, the square can be represented as Balls union (circle)



square covered with balls



every ball will have 2 balls around it

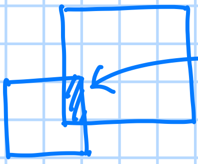
and as the balls are open, $\forall x \in$ Ball $\exists r > 0$ s.t $B_r(x) \subseteq$ Ball for this $B_r(x)$

we can have a smaller ball using x as center

Eg: ① Another basis for $(\mathbb{R}^2$, standard topology$)$ is $\mathcal{B}_1 = \{(a,b)\times(c,d) \mid a<b, c<d\}$

② Basis for $(\mathbb{R}^2, \| \ \|)$ is

$$\mathcal{B}_2 = \{(a, a+\varepsilon)\times(b, b+\varepsilon) \mid a,b \in \mathbb{R}, \varepsilon>0\}$$



can be written as union of squares

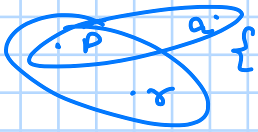
(a,b) $(a+\varepsilon, b)$

Note: In linear algebra $\{e_1 \dots e_{10}\}$ on $\mathbb{R}^{10}$ for any $x \in \mathbb{R}^2$
$x = \sum a_i e_i$ is s.t $(a_1, a_2 \dots a_{10})$ is unique,
this is not true here.

Def: (Sub Basis) A collection of open sets in (X, τ), we say
$\mathcal{C}$ is a subbasis for (X, τ) if (smaller than Basis)
$\mathcal{B} = \{\bigcap_{i=1}^{N} C_i \mid C_i \in \mathcal{C}, N \geq 1\}$ is a basis.

Exe: ① A basis is a subbasis of (X, τ) like
$\mathcal{B} = \{B(p, r) \mid p \in \mathbb{R}^2, r>0\}$ is Basis for $\mathbb{R}^2$.

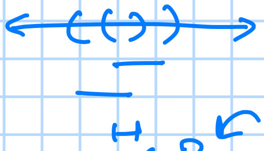
② $\tau = 2^X$, $C = \{\{p,q\} \mid p \neq q \in X\}$



$\{p,q\} \cap \{p,r\} = \{p\}$

$$B = \{\{p\} \mid p \in X\}$$

③ $X = \mathbb{R}$, $\mathcal{C} = \{(a, a+1) \mid a \in \mathbb{R}\}$



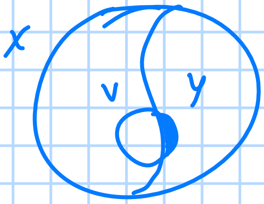
$\in B$ so all Balls in B

$$\mathcal{B} = \{(a,b) \mid a<b\}$$

17th Jan :

Subspace topology :

(Here τ_Y can/cannot have Y) ← Y does not have to be open

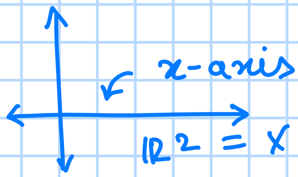


Let (X,τ) be a topological space and $Y \subseteq X$

then Y comes up with a natural topology τ_Y given by:

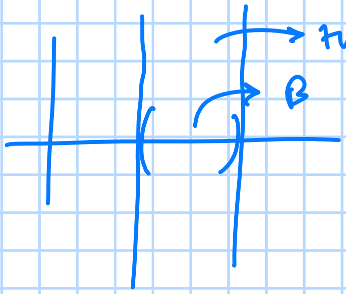
$$\tau_Y = \{B \subseteq Y \mid B = Y \cap V,\ \forall V \in \tau\}$$

Eg.



the $\tau_{x\text{-axis}} = \{B \subseteq Y \mid B = Y \cap V, \forall V \in \tau\}$

$= (\mathbb{R}, \text{standard topology})$



Note : V is not unique (yes, so diff V can have same B)

claim : τ_Y is a topology

proof : ① $\phi, Y \in \tau_Y$ as for $V = X$, $B = X \cap Y = Y$ as $Y \subseteq X$

$$\therefore B \in \tau_Y$$
$$\Rightarrow Y \in \tau_Y$$

also as $\phi \in \tau$
$$\Rightarrow \phi \cap Y = \phi \in \tau_Y$$

② $$\cup B_\alpha = \cup (Y \cap V_\alpha) = \underbrace{(\underset{\in \tau}{\cup V_\alpha}) \cap Y}_{\in \tau_Y} \quad \therefore \cup B_\alpha \in \tau_Y$$

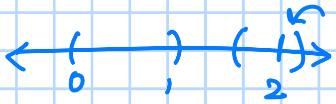
③ $$B_1 \cap B_2 = (Y \cap V_1) \cap (Y \cap V_2) = \underbrace{(Y) \cap \underset{\in \tau}{[V_1 \cap V_2]}}_{\in \tau_Y} \quad \therefore B_1 \cap B_2 \in \tau_Y$$

Eg :
① $X = \mathbb{R}$, $Y = [0,1]$

$\{(a,b), 0 < a < b < 1\} \cup \{[0, 1/2)\} \subseteq \tau_Y$

Exe: $Y = (0,1) \cap \{2\}$ $X = \mathbb{R}$, Find $A \subseteq Y$ s.t A is open, $2 \in A$

Ans:



$(2-\frac{1}{2}, 2+\frac{1}{2}) \in \tau$ ← general topology

then $(2-\frac{1}{2}, 2+\frac{1}{2}) \cap Y = \{2\} \in \tau_Y$

so $\{2\}$ is open in τ_Y

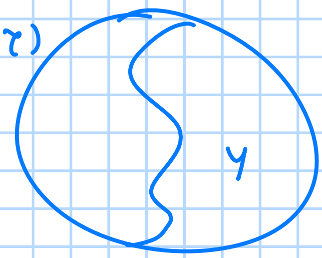
Theorem: If (X,d) is a metric space and $Y \subseteq X$ then
$\bar{d}(y_1, y_2) = d(y_1, y_2)$, $y_1, y_2 \in Y$
makes $(Y, \bar{d})$ a metric space

Eg:
$d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2}$
$\mathbb{R}^2$
x-axis dist
$d((x,0),(a,0)) = |x-a|$

How about the converse, suppose (Y, τ_Y) is a space and $Y \subseteq X$
Does there exist a topology τ_X on X so that τ_Y becomes the subspace topology (yes, if $\tau_Y \cup \{X\}$ is taken then this is also a topology)

Basis for subspace topology:

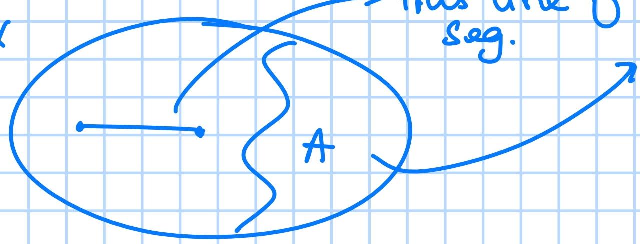
(X,τ)



suppose $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$
where $\mathcal{B}$ is basis for X, then
$\mathcal{B}'$ is a basis for τ_Y

Interior of a set in X:

X



this line seg. ∪ this set = A

A° = The largest open set in X which lies inside A

$= \bigcup_{\substack{V \in \tau \\ V \subseteq A}} V$ ← as topology is closed in union

Eg: ① $X = \mathbb{R}$, $A = (0,1)$, $A^\circ = (0,1)$

② $X = \mathbb{R}$, $A = [0,1]$, $A^\circ = (0,1)$ ← Here no $[0,\ldots)$ as if 0 then 0 in open set, so not in A

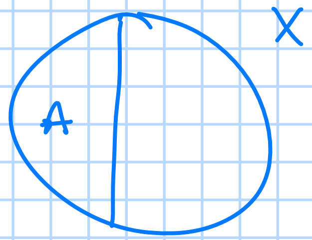
③ $X=\mathbb{R}$, $A=[0,1]\cup\{2\}$, $A^\circ=(0,1)$

④ $X=\mathbb{R}$, $A=\mathbb{Q}$ then $A^\circ=\phi$

⑤ $X=\mathbb{Z}$, $A=\{0\}$, $A^\circ=\{0\}$ ← as this is discrete topology

Closed set :

Def : A is closed if A^c or $X\setminus A$ is open in X.



Def : closure : Given $A\subseteq X$, closure of A, $\bar{A}$ = smallest closed set which contains A

Eg : ① $\bar{A}=[0,1]$

② $\bar{A}=A=[0,1]$

③ $\bar{A}=[0,1]\cup\{2\}$

④ $\bar{A}=\mathbb{R}$ as $A=\mathbb{Q}$

Claim : If $B\subseteq\mathbb{R}$, $B\supseteq\mathbb{Q}$ and B is closed, then $B=\mathbb{R}$

proof : suppose $\exists\, t_0\in\mathbb{R}$ s.t $t_0\notin B$

then $t_0\in \bigcup_{i\in I}(\alpha_i,\beta_i)=B^c$

but (α_i,β_i) contains infinite many rationals

$(\alpha_i,\beta_i)\cap\mathbb{Q}\neq\phi$

as $\mathbb{Q}\not\subseteq B^c$ as $\mathbb{Q}\subseteq B$

this is a contradiction

Eg : ⑤ $X=\mathbb{R}^2$, $A=\mathbb{Q}\times\mathbb{Q}$

$\bar{A}=\mathbb{R}\times\mathbb{R}$ or $\mathbb{R}^2$

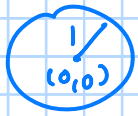
If B is closed in $\mathbb{R}^2$ and $(x_0,y_0)\notin B$

then B^c is open, $\exists\,\varepsilon>0$ s.t

$B_\varepsilon((x_0,y_0))\subseteq B^c$

but $(\mathbb{Q}\times\mathbb{Q})\cap B_\varepsilon((x_0,y_0))\neq\phi$ ✱

⑥ $A=B((0,0),1)$



$\bar{A}=\{(x,y)\mid x^2+y^2\leq 1\}$

7. Suppose (X,d) is a metric space and $p \in X$

$$A = B(p,1) = \{x \in X \mid d(x,p) < 1\}$$

maybe $\underbrace{\bar{A} = \{x \in X \mid d(x,p) \leq 1\}}_{\text{this is not correct}}$

as $X = \mathbb{Z}$ $\overset{\longleftrightarrow}{-1 \;\; 0 \;\; 1}$ $A = \{x \in \mathbb{Z} \mid |x| < 1\}$

$A = \{0\}$

$\bar{A} = \{0\}$

if $\bar{A} = \{x \in \mathbb{Z} \mid |x| \leq 1\} = \{-1, 0, 1\}$

as $\{-1,0,1\}$ is bigger then $\{0\}$, this is not a closure for $\{0\}$

Recap:

1. start with a space X
2. given $A \subseteq X$
3. $(A^o)^o = A^o$ open in X
4. $(\overline{A^o})$ is closed in X
5. $(\overline{A^o})^o$ is an open set

Note: $A \subseteq (\overline{A^o})^o$ is not true

for $X = \mathbb{R}$ $A = \mathbb{Q}$

$\mathbb{Q}^o = \phi$

now $(\bar{\phi})$ as ϕ is closed

$\bar{\phi} = \phi$

also $\phi^o = \phi$ as ϕ is open

Exe: Does $(\overline{A^o})^o \cdots$ ever stop?

$X = \mathbb{R}$, $A = [0,1]$

step 0: $A^o = (0,1)$

step 1: $\overline{A^o} = [0,1]$

$X = \mathbb{R}$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\}$

step 0: $A^o = \phi$

now as ϕ it repeats itself

$A = (0,1) \cup (1,2) \cup (2,3) \cup \cdots\cdots$

$A^o = A$

$\overline{A^o} = [0,\infty)$, $(\overline{A^o})^o = (0,\infty)$, $(\overline{A^o})^o = [0,\infty)$ Repeats

$X = \mathbb{R}$, $A = [0,1) \cup [2,3) \cup \cdots$

$A^\circ = (0,1) \cup (2,3) \cup \cdots$

$\overline{A^\circ} = [0,1] \cup [2,3] \cdots$

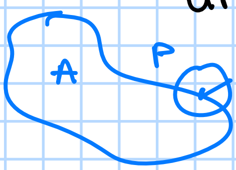
$(\overline{A^\circ})^\circ = (0,1) \cup (2,3) \cdots$

Q: $\forall A$, does it stop?

Q: What n do we stop / what step?

21st Jan:

Recall: $A \subseteq (X,d)$ then $\bar{A} := A \cup \{\text{All limit points of } A\}$ (in Real analysis)



limit point is x then $\forall r>0$, $(B(x,r)-\{x\}) \cap A \neq 0$

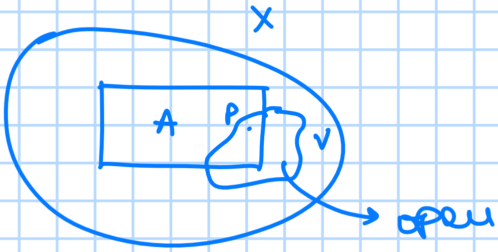
↑ infinitely many points in $B(x,r)$

(X,τ) given $A \subseteq X$, $\bar{A} :=$ smallest closed set in X which contains A.

Defn: A point $p \in (X,\tau)$ is called a limit point if given $p \in V$ open in X s.t $V \cap (A \setminus \{p\}) \neq \phi$

(This is for topology $p \in (X,\tau)$)

Note: A can be open/closed



lemma: For any $A \subseteq X$, $\bar{A} = A \cup \{\text{limit points of } A\}$

↑ (In topology it is the smallest closed set cont. A)

proof: let $A \cup \{\text{limit points of } A\} = B$

we want to show
① B is closed
② B is smallest closed set containing A

① B is closed if $X \setminus B$ is open:

let $p \in X \setminus B$ then

as $p \in X \setminus B$
$p \notin A$ and
$p \notin \{\text{limit point of } A\}$

so $\exists V$ s.t V is open in X and
$V \cap (A \setminus \{p\}) = \phi$
$\Rightarrow V \cap (A) = \phi$ as $p \notin A$

$\Rightarrow V$ is disjoint from A

$\forall p \in X \setminus B$, $\exists V_p$ (open in X) s.t

$V_p \cap A = \phi$
$\Rightarrow V_p \subseteq X \setminus B$
(as $\forall x \in V_p$, no x is a limit point as $V_p \cap A = \phi$)

so $X \setminus B = \bigcup_{p \in X \setminus B} V_p$

as V_p is open $\Rightarrow$ $X \setminus B$ is open
$\Rightarrow$ B is closed

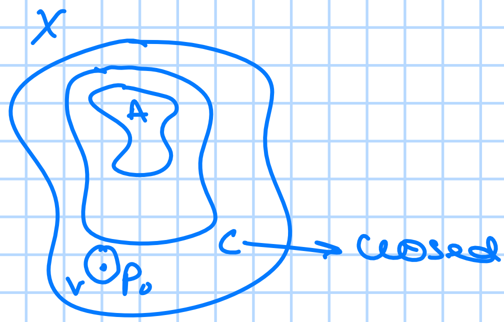
② Now we want to show B is smallest such which contains A.

To show: If $C \subseteq X$ s.t ① $A \subseteq C$
② C is closed
then C must contain B (i.e all limit points of A)

$\forall$ such C, $B \subseteq C$ $\Rightarrow$ B is smallest

suppose all limit points of A not in C
$\exists P_0 \notin C$ s.t

$P_0 \in$ limit of A



so $\exists P_0 \in V$ open s.t
(as C is closed)

$\Rightarrow V \cap C = \phi$
but as P_0 is a limit point for A

$\forall V \ni P$, $V \cap A \neq \phi$

as $V \cap C = \phi$
$\Rightarrow V \cap A = \phi$ ✱

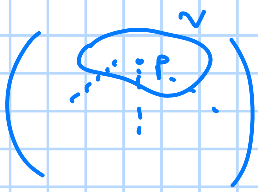
$\therefore$ $P_0 \in C$

$\Rightarrow$ all limit points of A in C

$\therefore$ $B \subseteq C$ $\forall$ such C ① closed
② contains A

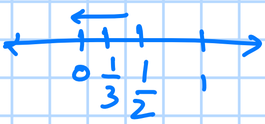
Defn: we say $x_n \in X$ and $P \in X$, $x_n \to P$ (x_n converges to P) if given any open set $P \in V$,

$\exists N = N(P, V)$ s.t $\in \mathbb{N}$
$x_N, x_{N+1}, \dots \in V$



Eg: ① $X = \mathbb{R}$, $x_n = \frac{1}{n}$

converging point : $P = 0$



for any open set V s.t $P \in V$
we have to prove
$\exists N = N(P, V)$ s.t
$x_N, x_{N+1}, \dots \in V$

we can do this for all Basis elements containing $P = 0$

$-\varepsilon \quad 0 \quad \varepsilon$ $\rightarrow$ $\frac{1}{N}, \frac{1}{N+1}, \dots \in (-\varepsilon, \varepsilon)$

$N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$ as $\frac{1}{N} < \varepsilon$

$-1 \quad 3$ we are only interested in $(-1,1)$

$-1 \quad 1$

② $X = \mathbb{Q}^c \cup \left\{ \frac{1}{n} \,\middle|\, n \geq 1 \right\}$

claim: here $\frac{1}{n}$ does not converge in X

proof: suppose $p \in X$ show $\frac{1}{n} \not\to p$ $\forall p \in X$

(as if $\frac{1}{n} \to p$ for some p and as $p \neq 0$ after some N it will be closer to 0)

$p \quad 0$ eventually all points will not be in a small ball about p

③ $X = \mathbb{R}$

$\tau =$ Topology generated by (By scenario-1) singleton 0

$\{\text{open in } \mathbb{R}\} \cup \{0\}$

$\mathcal{B} = \{$finite intersection of sets in $\{\text{open in } \mathbb{R}\} \cup \{0\})$

$\therefore$ Here $\frac{1}{n} \to p$ in (X, τ)

NO as $\{0\}$ is an open set

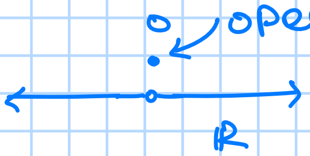
if $p = 0$ then

$\exists V = \{0\} \ni p$ $\rightarrow$ open

No such N s.t $\frac{1}{N} \notin \{0\}$ as for $\forall N$ $\frac{1}{N} > 0$ $\therefore \frac{1}{N} \notin \{0\}$

$\frac{1}{n} \not\to p$ for any $p \in \mathbb{R}$

as for other values, ② is used



Note : The above example, $\{0\}$ is open and was isolated

Eg: ④ $X = [0,2]$

$$\mathcal{B} = \{(a,b) \mid 0 < a < b < 2\} \cup \{[0,\varepsilon) \cup (2-\varepsilon, 2] \mid 0 < \varepsilon < \tfrac{1}{2}\}$$

let τ = topology generated by $\mathcal{B}$

Exe: The above τ, does $\frac{1}{n}$ converge in X?

so $\forall$ open set containing 0 it also contains 2.

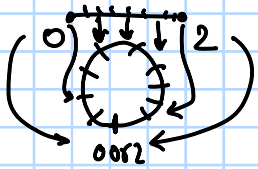
$\therefore$ $P = 2$ or 0, the open set $\exists N$ s.t

$N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$ s.t $\frac{1}{N}, \frac{1}{N+1}, \dots \in V$

but as $\forall$ open set cont 0
it also contains 2.

$\frac{1}{n} \to 0, 2$

Note: 0 and 2 are essentially the same



eg: ⑤ $X = \mathbb{R}$, $\tau = \{V \subseteq \mathbb{R} \mid V^c \text{ is finite}\} \cup \{\phi\}$

$x_n = \frac{1}{n} \to y$ for each $y \in \mathbb{R}$

as all open sets are very large

⑥ $X = \mathbb{R}$

$\tau = \{V \subseteq \mathbb{R} \mid V^c \text{ is countable}\} \cup \{\phi\}$

here for $\mathbb{R} \setminus \{1, \frac{1}{2}, \dots\} \in \tau$ and is open

for $p = 0$, $\nearrow$ is an open set

No N s.t.

$\frac{1}{N}, \frac{1}{N+1}, \dots \in V$

$\therefore \frac{1}{n} \not\to 0$, same for other points

Exe: How do we identify limit points of A? what we did above, using definition.

Def: X is called hausdorff if $\forall\, p, q \in X$, $p \neq q$ $\exists\, V_p$ and $V_q \in \tau$ s.t. $V_p \cap V_q = \phi$ where $p \in V_p$, $q \in V_q$

Lemma: If X is Hausdorff and $x_n \in X$, then $x_n \to x$ for atmost one x if any.

24th Jan :

Quiz-1 : Next thursday 7 – 8 pm

Syllabus : Material absorbed till (and including) today.

Exe : Find a sub-basis of $(\mathbb{R}, \tau)$ where $\tau = \{A \subseteq \mathbb{R} \mid A^c$ is countable $\}$

$$\mathcal{A} = \{A \subseteq \mathbb{R} \mid |A^c| \leq 10\}$$

$\mathcal{A}$ is not a basis as it should satisfy

① all open in topology $= \bigcup A_i$

② for $A_1 \cap A_2$, $\exists A_3 \in \mathcal{A}$ s.t $A_3 \subseteq A_1 \cap A_2$

here

$A_1 = \mathbb{R} \setminus \{1, 2, \ldots 10\}$

$A_2 = \mathbb{R} \setminus \{11, 12 \ldots 20\}$

so if $\mathcal{A}$ was a basis

$A_1 \cap A_2 = \bigcup C_i$

$|C_i| \leq 10$ but

$A_1 \cap A_2 = \mathbb{R} \setminus \{1, 2, \ldots 20\}$

where $|A_1 \cap A_2^c| = 20$

so not $\bigcup C_i$

it is not a subbasis also

$$C = \{A \subseteq \mathbb{R} \mid A^c \text{ is countable } \begin{array}{l} A^c \subseteq (0, \infty) \\ \text{or } A^c \subseteq (-\infty, 0) \end{array} \}$$

C is not a basis as

$$\underbrace{\mathbb{R} \setminus \{-1, 1\}}_{\in \text{ Topology}} = \underbrace{\bigcup_{\alpha \in I} A_\alpha}_{\text{Not Possible}}$$

if $A \in \tau$, A misses a few points

then

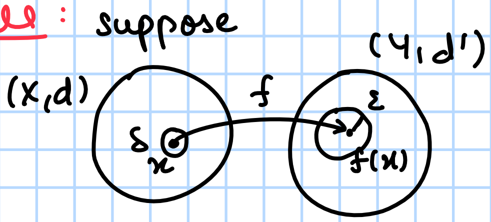
$A = A_1 \cap A_2 \to$ misses negative

$\hookrightarrow$ misses positive

$\to$ in C Both

Continuity :

Recall : suppose

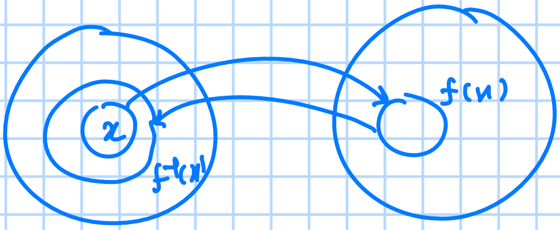


f is said to be continuous if given x, $\varepsilon > 0$

$\exists\, \delta_x = \delta(x, \varepsilon) > 0$ s.t

$d(x, z) < \delta_x \Rightarrow d'(f(x), f(z)) < \varepsilon$

Equivalently $f^{-1}(B(f(x), \varepsilon)) \supseteq B(x, \delta)$



$B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$

<u>Example</u>: Suppose $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 4(x)$

$\varepsilon = 1$ then what is δ

$x = 0$

$x=0 \quad |x - z| < \delta \Rightarrow |z| < \delta$

s.t $\quad 4|z| < \varepsilon$

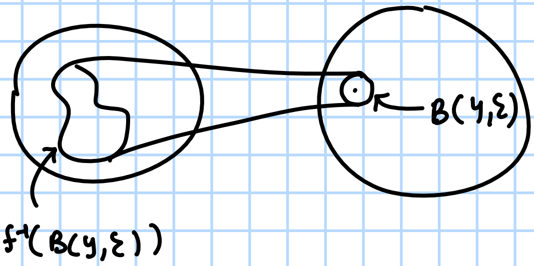
$\Rightarrow |z| < \varepsilon/4$

$\Rightarrow |z| < 1/4$

so $\delta < 1/4$

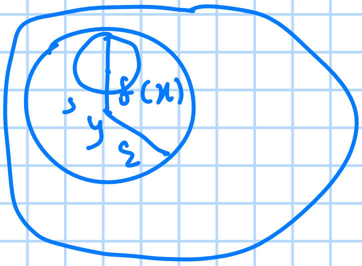
$f: X \to Y \quad f$ is cont

Now let's suppose $y \in Y$ and $\varepsilon > 0$, let's look at $f^{-1}(B(y,\varepsilon))$



claim: $f^{-1}(B(y,\varepsilon))$ is an open set in (X,d) given that f is cont

<u>proof</u>: let $x \in f^{-1}(B(y,\varepsilon))$ then $d'(f(x),y) < \varepsilon$



if $s = d'(f(x),y)$ then, by triangle inequality

$\varepsilon - s = r$

$B(f(x), \varepsilon - s) \subseteq B(y, \varepsilon)$

as $\forall z \in B(f(x), \varepsilon - s)$

$d(z, f(x)) < \varepsilon - s$

$d(z,y) < d(z, f(x)) + \underbrace{d(f(x), y)}_{= \varepsilon}$

$\Rightarrow z \in B(y,\varepsilon)$

so $B(f(x), \varepsilon - s) \subseteq B(y,\varepsilon)$

so, $f^{-1}(B(f(x), \varepsilon - s)) \supseteq B(x,\delta)$

By continuity

$B(x,\delta) \subseteq f^{-1}(B(f(x), \varepsilon - s))$

$\subseteq f^{-1}(B(y,\varepsilon))$

$\Rightarrow B(x,\delta) \subseteq f^{-1}(B(y,\varepsilon))$

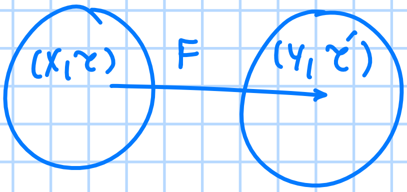
$$\left(f^{-1}(B(y,\varepsilon)) = \bigcup_{d(f(x),y)<\varepsilon} B(x,\delta_x) \right)$$

$\forall x \in f^{-1}(B(y,\varepsilon)), \exists B(x,\delta) \subseteq f^{-1}(B(y,\varepsilon))$

so, $f^{-1}(B(y,\varepsilon))$ is <u>open</u>.

Note: The above proof also tells us that if U is open in Y then $f^{-1}(U)$ is open in X.

Def: (continuity in τ) Suppose $f: X \longrightarrow Y$



f is called continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

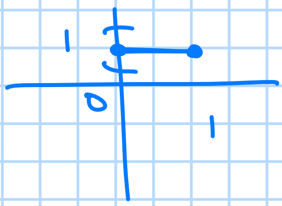
lemma: If $\mathcal{B}_Y$ is a basis for (Y, τ') so that for each $B \in \mathcal{B}_Y$ $f^{-1}(B) \in \tau$, then f is cont.

proof: As $\mathcal{B}_Y$ is a Basis

$$U = \text{open in } Y = \bigcup_{\alpha \in I} \underset{\in \mathcal{B}_Y}{U_\alpha}$$

$$\Rightarrow f^{-1}(U) = \underbrace{\bigcup_{\alpha \in I} \underbrace{f^{-1}(U_\alpha)}_{\text{open in } X}}_{\text{open in } X}$$

Eg: ① $f(x) = \begin{cases} 1 & : 0 \leq x \leq 1 \\ 0 & : \text{otherwise} \end{cases}$



$f^{-1}\left(\frac{1}{2}, \frac{3}{2}\right) = [0, 1]$

② given any X, Y choose $y_0 \in Y$ Define $f: X \longrightarrow Y$

$f(x) \equiv y_0$

for every open set from Y

$f^{-1}(\underbrace{Y \ni y_0}_{\text{open}}) = X \longrightarrow \text{open}$ or $\phi \rightarrow \text{open}$ $(\phi, X \in \tau)$

so f is cont

③ $f: \mathbb{Z} \longrightarrow Y$

$\to$ every set is open

so

$f^{-1}(\text{open set}) = \text{open}$

$\therefore$ any f is cont here

Exe: $f: X \to Y$

Is it possible to construct a topology τ on X s.t

$f: (X, \tau) \longrightarrow (Y, \tau')$

Yes, $\tau = 2^X \to$ all subsets of X are open

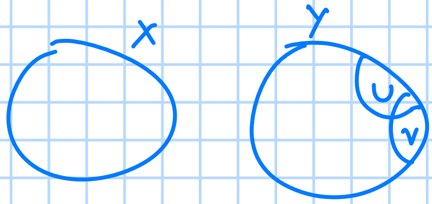
Exe: Does there exist the "smallest topology" on X that makes f a continuous map?

$$\tau = \{ f^{-1}(U) \mid U \text{ is open in } Y \}$$

claim is that τ forms a topology on X

① $\phi = f^{-1}(\phi)$

$X = f^{-1}(Y)$



② $A = f^{-1}(U)$

$B = f^{-1}(V)$

$$A \cap B = f^{-1}(U) \cap f^{-1}(V)$$
$$= f^{-1}(\underbrace{U \cap V}_{\text{open in } Y})$$

$\Rightarrow A \cap B$ is open in X

③ similarly $A_1 = f^{-1}(U_1)$

$A_2 = f^{-1}(U_2)$

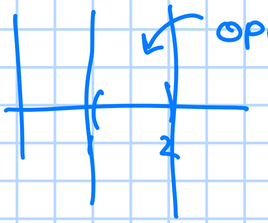
$\vdots$

$$\Rightarrow f^{-1}(U_1) \cup f^{-1}(U_2) \cdots = f^{-1}\Big(\bigcup_{\alpha \in I} U_\alpha\Big)$$
$$= A_1 \cup A_2 \cup \cdots$$

as $\bigcup_{\alpha \in I} U_\alpha$ is open in Y.

Eg: $f : \mathbb{R}^2 \to (\mathbb{R}$, standard topology$)$

$f(x, y) = x$



open set in topology which makes a continuous function (f becomes cont)

31^{st} Jan:

Def: (Basis) Suppose X is a space, A collection of open set in X, say $\mathcal{B}$ is called basis if each open set V in X can be written as

$$V = \bigcup_{B \in \mathcal{B}} B$$

this will be true as Basis element also open in τ

$$B_1 \cap B_2 = \bigcup_{\alpha \in \mathcal{B}} B_\alpha$$

Note: Not in def of basis:

Suppose X is a set $\mathcal{B} = \{B_\alpha \subseteq X \mid \alpha \in \Sigma\}$ is a collection of subsets of X which satisfies that

$$\left.\begin{aligned} B_1 \cap B_2 &= \bigcup_{B \in \mathcal{B}} B \\ X &= \bigcup_{B \in \mathcal{B}} B \end{aligned}\right\} \text{way of constructing basis}$$

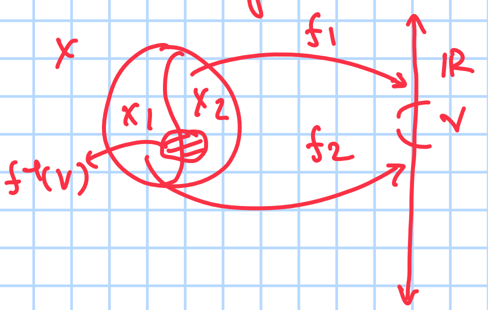
Now $\tau := \left\{ \bigcup_{\alpha \in \Sigma} B_\alpha \mid B_\alpha \in \mathcal{B} \right\}$ is a topology on X for which by construction $\mathcal{B}$ is a basis

Counter example: $\mathcal{B} = \{\mathbb{R}, [0,2], [1,4]\}$

$\tau = \{\phi, \mathbb{R}, [0,2], [1,4]\} \longrightarrow$ not a topology

Theorem: constructing continuous functions from smaller pieces:

X_1, X_2 open in X (subset topology)



define $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} f_1(x), & x \in X_1 \\ f_2(x), & x \in X_2 \end{cases}$$

then $f: X \to \mathbb{R}$ is also cont

proof:

$V \in \mathbb{R} \to$ open in $\mathbb{R}$

$$f^{-1}(V) = (f^{-1}(V) \cap X_1) \cup (f^{-1}(V) \cap X_2)$$

$$= \underbrace{(f_1^{-1}(V) \cap X_1)}_{\text{open in } X_1} \cup \underbrace{(f_2^{-1}(V) \cap X_2)}_{\text{open in } X_2} \Rightarrow f^{-1}(V) \text{ is open}$$

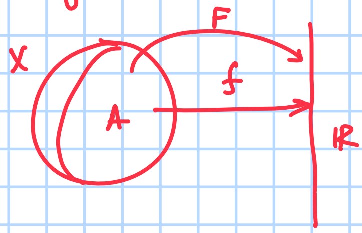
for X_1 subset topology

Theorem: If $X = \bigsqcup_{\alpha \in I} X_\alpha$ $f_\alpha : X_\alpha \to \mathbb{R}$ cont then $f: X \to \mathbb{R}$

$f(x) = f_\alpha(x)$ $\forall x \in X_\alpha$ is also continuous

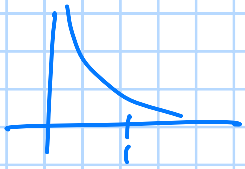
proof: similar to above, but we use the fact that we can take arbitrary union of open sets in τ.

Exe : Suppose $f : A \to \mathbb{R}$ cont



Does there exist a cont $F : X \to \mathbb{R}$ s.t. $F|_A = f$

$f : (0,1) \to \mathbb{R} \to$ NO



if $f(x) = \frac{1}{x}$ then cont on $(0,1)$

not for whole $\mathbb{R}$

↓

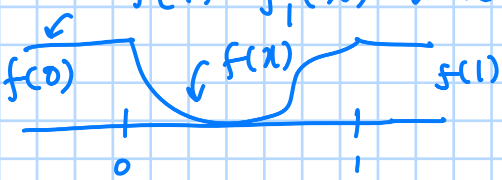
for any function as it blows up

Suppose $f : [0,1] \to \mathbb{R} \to$ yes :

as extension

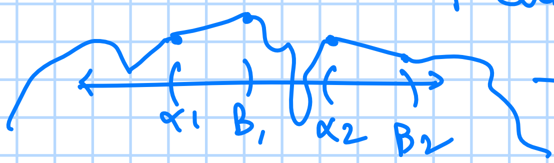
$f(0) = f_0(x) \ \forall x \in (-\infty, 0)$

$f(1) = f_1(x) \ \forall x \in (1, \infty)$



A is closed subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ continious

as $A^c = \bigsqcup_{i \in I} (\alpha_i, \beta_i)$ → union of disjoint intervals

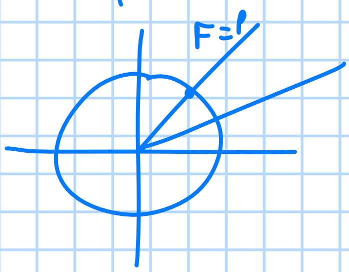


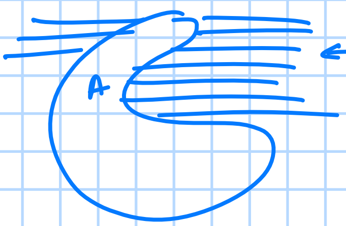
→ this is an obvious choice

Let : $A \subseteq \mathbb{R}^2$ closed , $f : A \to \mathbb{R}$ continious

$A = \{(x,y) \mid x^2 + y^2 < 1\}$ yes

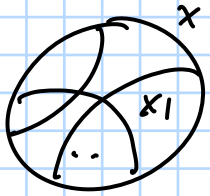
$f : A \to \mathbb{R}$





← if A is closed something can be made for $\mathbb{R}^2$

So far : Each X_i is open



$$f_i\big|_{X_i\cap X_j} = f_j\big|_{X_i\cap X_j}$$

then $\exists\, F: X \to \mathbb{R}$
s.t
$F|_{X_i} = f_i$

suppose each X_i is closed in X ← Closed

$$X = \bigcup_{i=1}^{N} X_i \text{ and}$$ ← Finite

(Here finite as finite union of closed sets is finite, not same in arbitrary case)

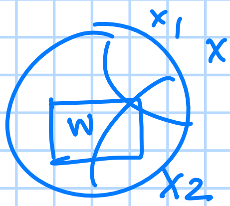
$f_i : X_i \to \mathbb{R}$ continious

$$f_i\big|_{X_i\cap X_j} = f_j\big|_{X_i\cap X_j} \quad \text{then } \exists\, F: X \to \mathbb{R} \text{ cont}$$
$$F|_{X_i} = f$$

lemma : $f: X \to Y$ is continious $\Leftrightarrow$ $f^{-1}(W)$ is closed in X for W closed in Y

proof : $(f^{-1}(W))^c = f^{-1}(W^c)$ ← open
$\vdash$ open

$\forall x \in f^{-1}(W^c)$
$\Leftrightarrow f(x) \in W^c$
$\Leftrightarrow f(x) \in X\setminus W$
$\Leftrightarrow f(x) \notin W$
$\Leftrightarrow x \notin f^{-1}(W)$
$\Leftrightarrow \forall x \in (f^{-1}(W))^c$



$$F^{-1}(W) = \bigcup_{i=1}^{N} \underbrace{[F^{-1}(W)\cap X_i]}_{\text{closed}}$$ ← finite

$\Rightarrow F^{-1}(W)$ is closed in X

Remark : Then fact that N is finite is important as:

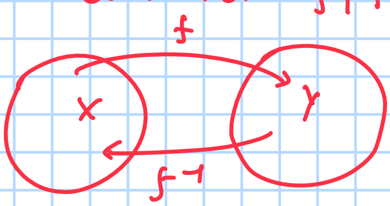
$$\mathbb{R} = \bigsqcup_{x\in\mathbb{R}} \{x\}$$

(one more $[1, \frac{1}{n}]$ .... and $\{0\}$)

given $F: \mathbb{R} \to \mathbb{R}$
$f_x : \{x\} \to \mathbb{R}$
$f_x(x) = F(x)$
so every f^n on $\mathbb{R}$ is cont, this does not make sense.

Def : (Homeomorphism) $f: X \to Y$ is called homeomorphism

$f: X \to Y$ is called homomorphism if f is bijective and both f, f^{-1} is cont



① f is bijective
② f, f^{-1} is cont

eg: $\{1,2,\ldots,N\} \longleftrightarrow \{N, N+1, \ldots, 2N\}$

$(0,1) \cong (2,3)$

$(0,1) \cong (1,\infty)$

4th Feb :

Eg: Are $[0,1)$ and $(0,1)$ homeomorphic

so $\exists F$ s.t

F is ① Bijective

② F and F^{-1} are cont

now $\exists F$ s.t $(0,1) \longrightarrow [0,1)$

which is bijective

$\{1,2,3,\ldots\} \to \{0,1,2,3,\ldots\}$

we show same points by shifting

$f(n) = n-1$ show f is one-one onto

now $A = \{x_1, x_2, \ldots\} \subseteq (0,1)$

$$f(x) = \begin{cases} 0 & x = x_1 \\ x_{i-1} & x = x_i, \ i > 1 \\ x & \text{on } (0,1)\setminus A \end{cases}$$

But they are not homeomorphic:

if $f:(0,1) \longrightarrow [0,1)$ is bijective

then f has to be strictly inc or dec

wlog inc then if $f(p) = 0$

then for $x \in (0,p)$

$f(x) < 0$ $*$

$\therefore$ not possible

$\therefore$ f is not homeomorphic

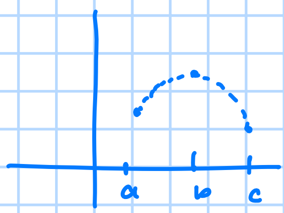
Note: Bijective and cont f then f is either strictly inc or decreasing

proof of this is from IVT

as suppose $\exists$ $a,b,c \in (0,1)$ s.t

$f(a) < f(b)$

and $f(b) > f(c)$



claim: $(f(a), f(b)) \subseteq f((a,b))$ (IVT)

proof: if $(f(a), f(b)) \not\subseteq f((a,b))$

then

$\exists\, \alpha \in (f(a), f(b))$

s.t

$f(a) < \alpha < f(b)$

$f(x) \neq \alpha \ \ \forall x \in (a,b)$

take

$A = \{x \mid f(x) < \alpha\} \subseteq (a,b)$

A is non-empty

as $a \in A$

also, as A is bounded

$\Rightarrow \exists\, x_0 = \sup A$

now $f(x_0^-) < \alpha$
and
$f(x_0^+) > \alpha$ } jump in cont ✗

as $f(x_0) < \alpha$ choose $\varepsilon > 0$ s.t
$f(x_0) + \varepsilon_0 < \alpha$
but f is cont so, $\exists \delta_0$ s.t
(this is just the proof of IVT)

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

on $(x_0, x_0+\delta)$
$\longrightarrow f < f(x_0) + \varepsilon < \alpha$
but by defination of x_0 this is not possible

Quotient topology:

Recall: X is a set, $f: X \longrightarrow \mathbb{R}$

$$\tau_f = \{ f^{-1}(V) \mid V \text{ is open in } \mathbb{R} \}$$

↑ topology generated by f
↓ this makes f cont in τ_f

so that,
$f : (X, \tau_f) \longrightarrow (\mathbb{R}, \text{standard topology})$
becomes continious

Exe: what about the other way? i.e suppose X is a topology, Y is a set $f: X \longrightarrow Y$
we want to define topology on Y so f becomes cont in X.
Is it possible to find the largest topology s.t this happens?

one such topology $\tilde{\tau}$ on Y, then it must happen that

$$\tilde{\tau} \subseteq \{ V \mid f^{-1}(V) \text{ is open in } X \}$$

$\{ V \mid f^{-1}(V) \text{ is open in } X \}$ is a topology on Y

Claim: $\{ V \mid f^{-1}(V) \text{ is open in } X \}$ is a topology on Y

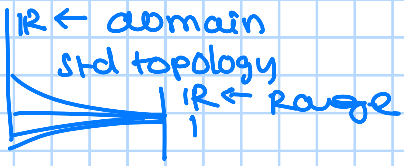
proof: $f^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha f^{-1}(V_\alpha)$

$\bigcup_\alpha V_\alpha \in \tau$

$f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2)$

$\Rightarrow V_1 \cap V_2 \in \tau$

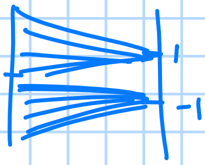
Eg: $f: \mathbb{R} \longrightarrow \mathbb{R}$



$$\tilde{\tau}_f = \{ V \mid f^{-1}(V) \text{ is open} \}$$
$$= \{ \emptyset, \{1\}, \mathbb{R} \} \cup \{ A : 1 \notin A \} = 2^{\mathbb{R}}$$

Eg: $f(x) = \begin{cases} 1 & ; 0 \leq x < \infty \\ -1 & ; x < 0 \end{cases}$

$(\mathbb{R}, std) \longrightarrow \mathbb{R}$



lets take cases

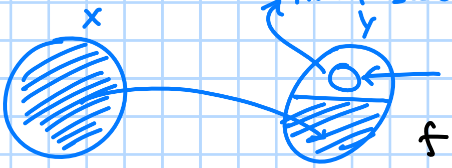
if A : 1   -1

1	-1		
✗	✗	then	$f^{-1}(A) = \phi$
✗	✓	then	$f^{-1}(A) = (-\infty, 0)$
✓	✗	then	$f^{-1}(A) = [0, \infty) \leftarrow$ not open
✓	✓	then	$f^{-1}(A) = \mathbb{R}$

$$\tilde{\tau}_f = \{ A \mid A \cap \{-1, 1\} = \phi \} \cup \{ A \cap \{1, -1\} = \{-1\} \} \cup \{ A \cap \{1, -1\} = \{1, -1\} \}$$

In other words,

V is open in $(Y, \tilde{\tau}_f)$ iff $f^{-1}(V)$ is open in X



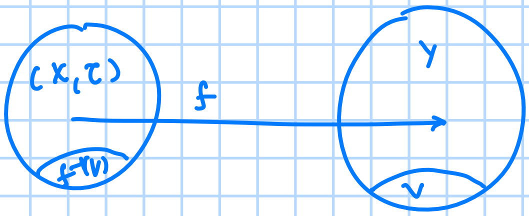
Any set here is open

Any open set

f is not surjective so there are many open sets in Y

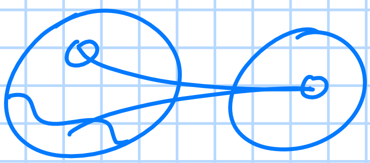
Note: we will assume f is surjective to construct $\tilde{\tau}_f$

Def: Suppose $f: X \longrightarrow Y$ surjective b/w two spaces X, Y. f is called a quotient map if $\tau_Y = \tilde{\tau}_f$ and τ_Y is called quotient topology



what about $\tau_f \supseteq \{ f(\omega) \mid \omega \text{ is open in } X \}$ $\rightarrow$ topology generated by this

this is not a good choice for topology as:

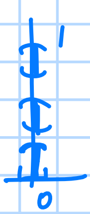


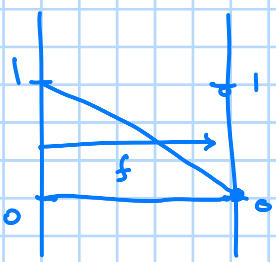
a lot of things get mapped to same set as f is not one-one

Eg: Quotient maps $\tilde{\tau}_f = \tau_Y$

$$f : ([0,1], \text{standard}) \longrightarrow [0,1)$$

$$f(t) = \begin{cases} t & t \neq 1 \\ 0 & t = 1 \end{cases}$$



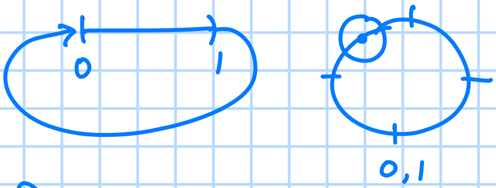


$$\tilde{\tau}_f \supseteq \{ (a,b) \mid a > 0, b < 1 \} \cup \{ [0, \varepsilon) \cup (1-\varepsilon, 1) \mid 0 < \varepsilon < 1/3 \}$$

Note: (X, d) metric space then:

$$\mathcal{B} = \{ B(x,r) \mid x \in X, r < 1 \} \text{ is also a basis}$$

Exe: $([0,1), \tilde{\tau}_f) \cong S^1$



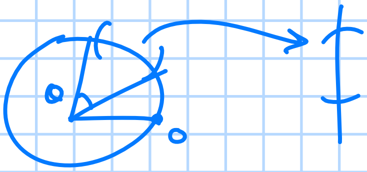
$f(t) = e^{2\pi i t} \quad 0 \leq t < 1$
$= (\cos 2\pi t, \sin(2\pi t))$
is homeomorphism

① injective: $e^{2\pi i t_1} = e^{2\pi i t_2}$
$t_1 = t_2$

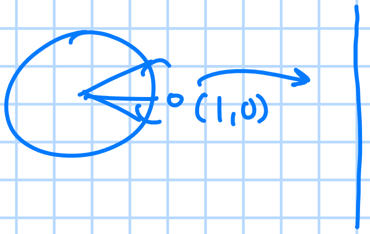
② also $\forall \; e^{2\pi i t} \in S^1$
$\exists \, t \Rightarrow f(t) = e^{2\pi i t}$

$\therefore f$ is bijective

③ f is continuous as:



no problem in continuity if 0 not contained



$(1,0) \in U = \{ e^{2i\theta} : 0 \leq \theta < \varepsilon, 1-\varepsilon < \theta < 1 \}$

$f^{-1}(U) = \{ \theta \mid 0 \leq \theta < \varepsilon, 1-\varepsilon < \theta < 1 \}$
$= [0, \varepsilon) \cup (1-\varepsilon, 1)$

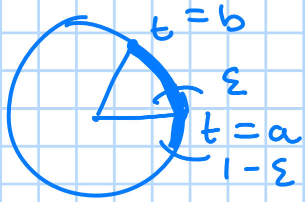
f^{-1} is also cont

$f(t) = e^{2\pi i t}$

$f(U)$ open for all U open
then
$f^{-1}(U)$ is cont

now, $\tilde{\tau}_f$ = generated by $\{ (a,b) \mid a > 0, b < 1 \}$
$\cup \{ [0, \varepsilon) \cup (1-\varepsilon, 1] \mid \varepsilon < 1/3, \varepsilon > 0 \}$

$U \leftarrow$ open



for $\therefore$ open in S^1

$\therefore f^{-1}$ is cont

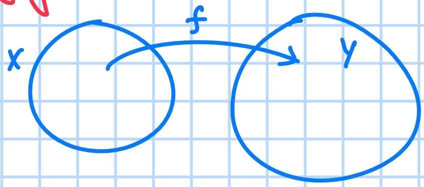
so by ① f is bijective
② f, f^{-1} is cont $\Rightarrow ([0,1), \tilde{\tau}_f) \cong S^1$

7th Feb :

Recap: Quotient topology and quotient map
we have

- A topological surface (X, τ)
- $f : X \to Y$, Y has not topology yet
  we want τ_Y s.t $f^{-1}(U)$ open in X for U open in Y

Can we create a topology on Y s.t f becomes cont ?

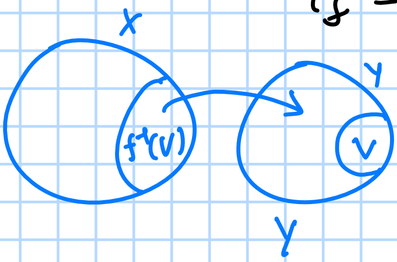


there is a trivial topology $\{\phi, Y\}$ but we want to see largest topology on Y.

Can we create the largest topology on Y s.t f is cont?
yes, this topology is called Quotient topology

$$\tilde{\tau}_f = \tau_Y$$

$$\tilde{\tau}_f = \{ U \subseteq Y \mid f^{-1}(U) \text{ is open in } X \}$$



f is called Quotient map

Why cannot we take $\tilde{\tau}_p$ = all subsets of Y ?

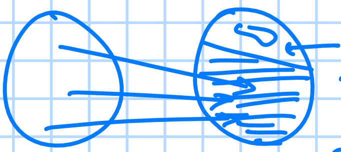
As not for all functions f might be cont as there can be a weird looking set in Y s.t $f^{-1}(U)$ is <u>not</u> open

<u>Eg</u> :

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x$
if $\tau_Y = 2^{\mathbb{R}}$ then

$[0,1]$ open in Y
but $f^{-1}[0,1] = [0,1]$ is closed in

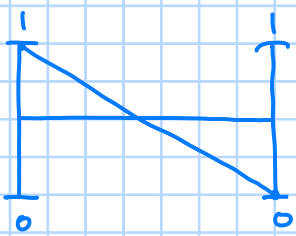
Why do we require f to be surjective ?



Any set here is an open set as $f^{-1}(U) = \phi$

there will be many open sets

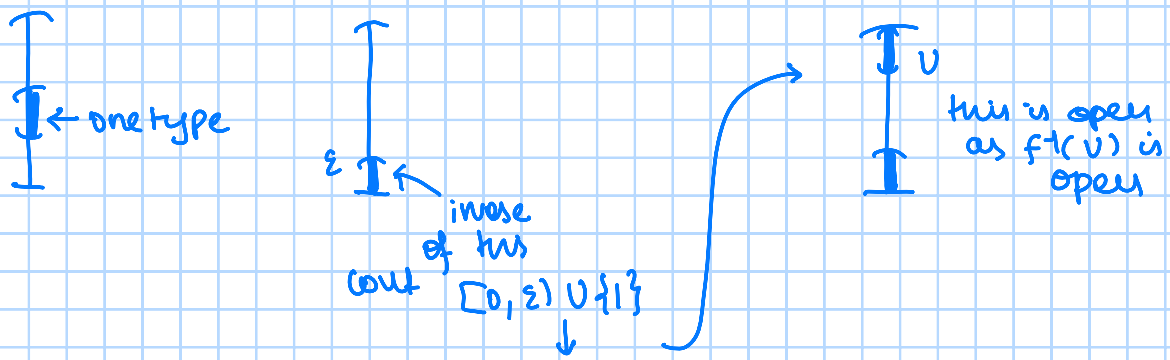
<u>Eg</u> :

$X =$ 

$f : X \to Y$

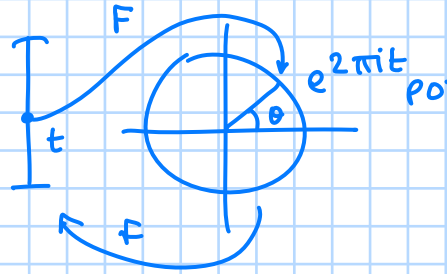
$Y = [0,1)$ $f(x) = \begin{cases} 0 & ; x = 1 \\ x & ; x \neq 1 \end{cases}$

two types of open set in $([0,1), \tilde{T}_f)$



Claim: The following $F: [0,1) \longrightarrow S^1$ is homeomorphism where $F(t) = e^{2\pi i t}$

proof:

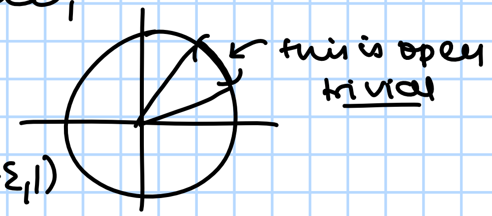


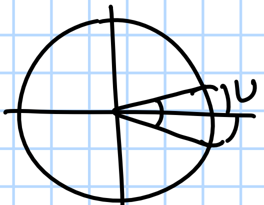
① F is bijective (By construction)

② F is cont:

$F^{-1}(U)$ open in $\tilde{T}_X$ for U open in $\tilde{T}_Y$.

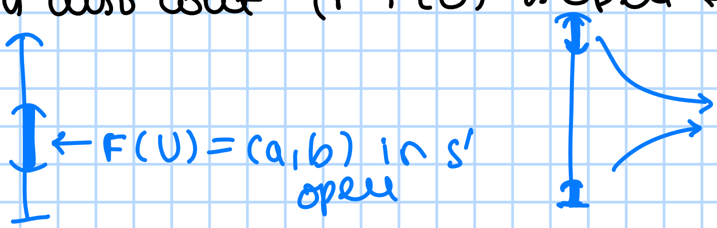
now,



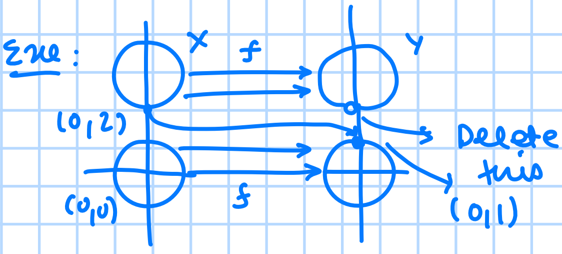


$f^{-1}(U) = [0,\varepsilon) \cup (1-\varepsilon, 1)$

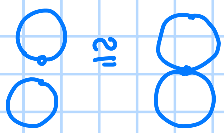
③ F^{-1} is also cont $(F^{-1})^{-1}(U)$ is open $\Leftrightarrow$ $F(U)$ is open as F is bijective



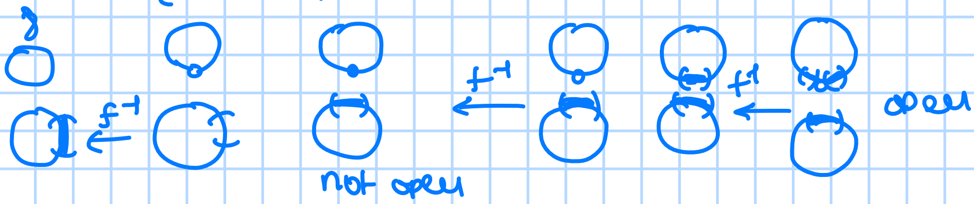
Exe:



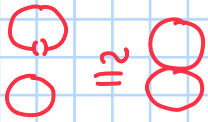
$$f(x,y) = \begin{cases} (x,y) & ; (x,y) \neq (0,2) \\ (0,1) & ; (x,y) = (0,2) \end{cases}$$



$Y = \{(x,y) \mid x^2+y^2 = 1\} \cup \{(x,y) \mid x^2 + (y-3)^2 = 1\} \setminus \{(0,2)\}$

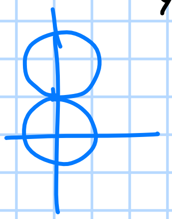


claim:



proof: $F: Y \to Z$

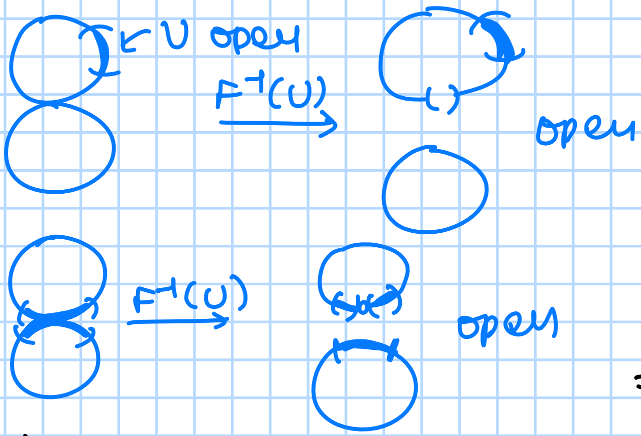
$$Y = \underbrace{\{(x,y) \mid x^2+y^2=1\}}_{Y_1} \cup \underbrace{\{(x,y) \mid x^2+(y-3)^2=1\}}_{Y_2} \setminus \{(0,2)\}$$



$$F(x,y) = \begin{cases} (x,y) & \text{in } Y_1 \\ (x,y-1) & \text{in } Y_2 \end{cases}$$

① F is bijective is trivial

② F^{-1} is cont as



By defination open

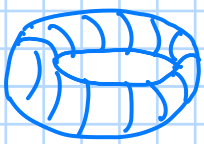
$F^{-1}(U)$ is open for U op

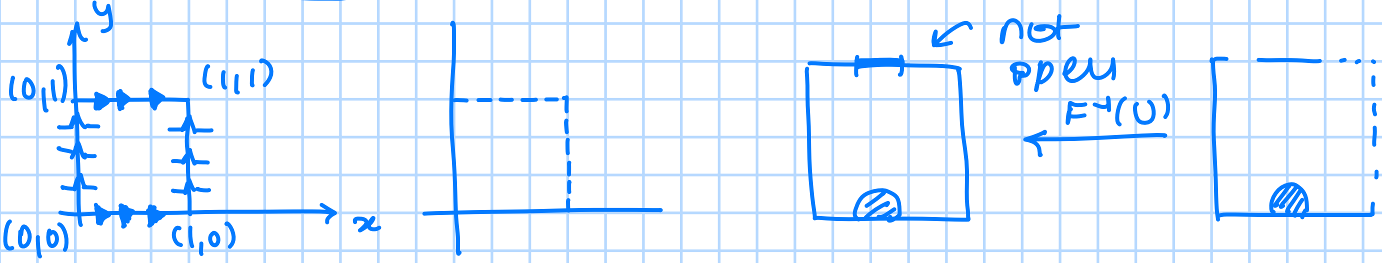
$\Rightarrow$ F is cont (While proving show $F^{-1}(B) \in \mathcal{B}_X$)

similarly F^{-1} is cont

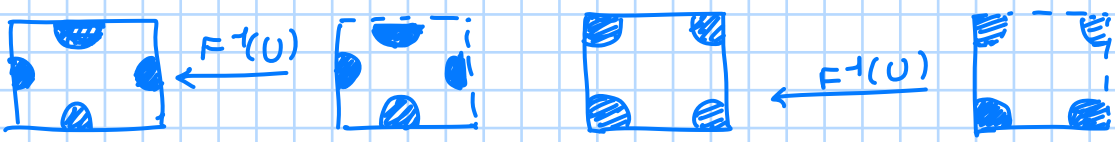
Exe:

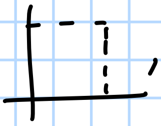
$$S^1 \times S^1 = \{(x,y,z,\omega) \in \mathbb{R}^4 \mid x^2+y^2 = z^2+\omega^2 = 1\}$$

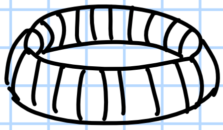




$$F(x,y) = \begin{cases} (x,y) & ; (x,y) \in C \\ (0,y) & ; x=0, y \neq 1 \\ (0,y) & ; x=1, y \neq 1 \\ (x,0) & ; y=1 \end{cases}$$



Note :  $/\tilde{\sim}_f \cong S^1 \times S^1$



$$S^1 \times S^1 = \left\{ (x,y,z,\omega) \in \mathbb{R}^4 \,\middle|\, x^2+y^2 = 1 = z^2+\omega^2 \right\}$$

$$F = (e^{2\pi i t}, e^{2\pi i s})$$

11th Feb :

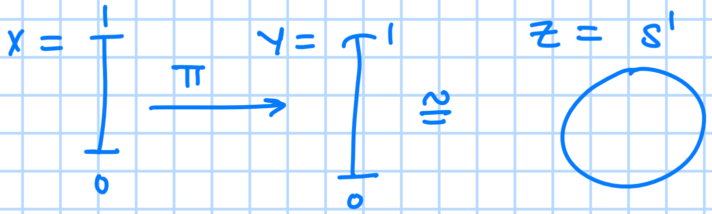
Quiz-2 : Next Thursday, Feb 20, syllabus : From that is not cover in Quiz-1 to materials covered till this friday (14th Feb)

goal : we have space X, we also have another space Z, we want a Quotient map (find π and Y for given X)

$\pi : X \longrightarrow Y$

so $(Y, \tilde{\tau}_\pi)$ is homeomorphic to Z.

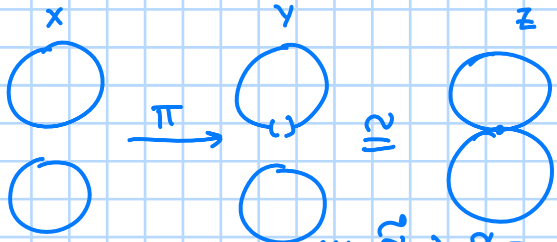
Exe :



$(Y, \tilde{\tau}_\pi) \cong S^1$

set ↳ topology on Y

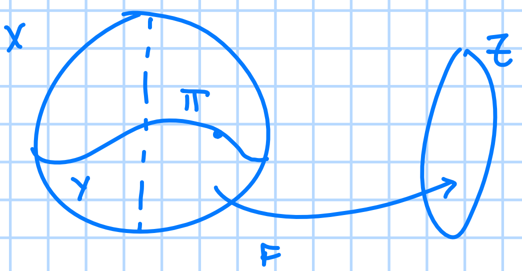
Exe :

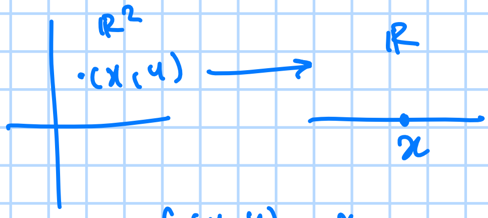


$(Y, \tilde{\tau}_\pi) \cong Z$

Theorm : suppose X, Z are topological spaces and $f : X \longrightarrow Z$ is a quotient map then $\exists\, Y \subseteq X$ (just as a subset) and $\pi : X \longrightarrow Y$ so $(Y, \tilde{\tau}_\pi) \cong Z$

($f : X \longrightarrow Z$ ↳ quotient map means Z sits inside X, f is surjective)





$f(x, y) = x$

Claim : $f(x,y) = x$ $\quad \exists$ such $Y \subseteq \mathbb{R}^2$

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

(this is just a small example)

proof :

first is f a quotient map :

① f is surjective

② $f^{-1}(U)$ open $\Leftrightarrow$ U open

(defination of Quotient map)

$(\tilde{\tau}_f = \{ U \mid f^{-1}(U) \text{ is open} \})$

① $\forall\, x \in \mathbb{R} \Rightarrow \exists (x, 0)$ s.t $f(x, 0) = x$

② for $(a, b) \subseteq \mathbb{R}$

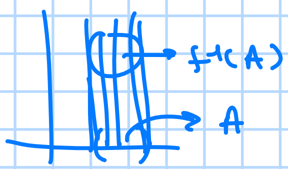
↳ open

$f^{-1}((a,b)) = \{ (x,y) \mid a < x < b,\ y \in \mathbb{R} \}$

$f^{-1}(A)$ open then

if $(a, b) \in f^{-1}(A)$

then $a \in A \Rightarrow A$ is open



↳ Becomes whole strip

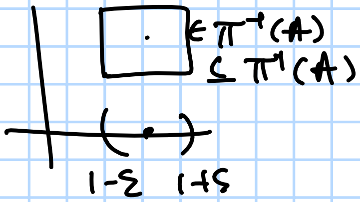
take $Y = \{(x,0) \mid x \in \mathbb{R}\}$

define
$F : Y \longrightarrow Z$
$F(x,0) = x$

$\pi : X \longrightarrow Y$
$\pi(x,y) = (x,0)$
then $(Y, \tilde{\tau}_\pi) \cong Z$

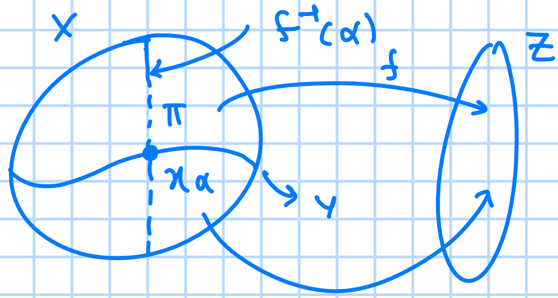
lemma: $\pi^{-1}(A)$ is open in $\mathbb{R}^2$ then A is open in (Y, subspace topology)

$1 \in A$, $(1,1) \in \pi^{-1}(A)$



(this is to show subspace topology $\equiv$ Quotient topology)

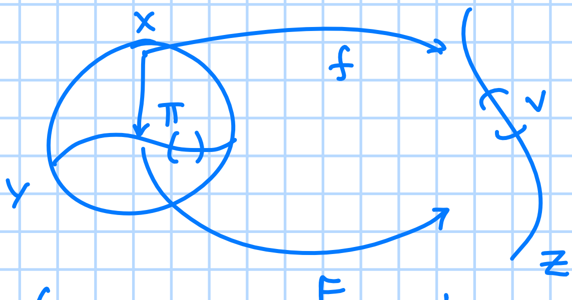
proof: For $\forall \alpha \in Z$ look at $f^{-1}(\alpha) \subseteq X$
choose $x_\alpha \in f^{-1}(\alpha)$
let $Y = \{ x_\alpha \mid \alpha \in Z \}$
define $F(x_\alpha) = f(x_\alpha)$
define $\pi : X \longrightarrow Y$ s.t. $\pi|_{f^{-1}(\alpha)} = x_\alpha$



F is homeomorphic

(π is a Quotient map as ① is trivial, ② is also trivial)

now our claim is that $F : (Y, \text{quotient}) \longrightarrow Z$ is homeomorphism



$f = F \circ \pi$ $\Big(F(x_\alpha) = f(x_\alpha)$ $\pi|_{f^{-1}(\alpha)} = x_\alpha$

also for V open $\subseteq Z$ $F \circ \pi|_{f^{-1}(\alpha)} = F(x_\alpha) = f(x_\alpha) \Big)$

$F^{-1}(V)$ should be open in Y

$F^{-1}(V)$ open in $Y \Leftrightarrow \pi^{-1}(F^{-1}(V))$ is open in X
(π is Quotient map)

$f = F \circ \pi$
$\pi^{-1}(F^{-1}(V)) = f^{-1}(V)$ which is open in X

thus F is cont ($F^{-1}(V)$ open $\Leftarrow \pi^{-1}F^{-1}(V)$ open $\Leftarrow f^{-1}(V)$ open $\Leftarrow V$ open)

now show U open $\Rightarrow F(U)$ is open (F is open map)

If W is open in Y then
we have to show F(W) is open

as $f: X \longrightarrow Z$
if $f^{-1}(F(W))$ is open in X then by defination of Quotient map $F(W)$ open

now, W is open in (Y, quotent)

$\pi^{-1}(W)$ open in X

$\pi^{-1}(W) = f^{-1}F(W)$

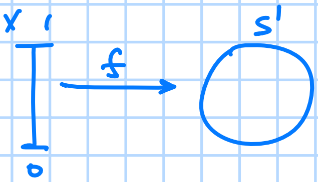
as $f = F \circ \pi$

$f \circ \pi^{-1} = F$

$\pi^{-1} = f^{-1} \circ F$

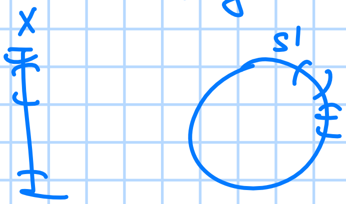
I give you X, and S^1 give example of quotient map on X, so that resulting quotient space is homeomorphic to Z

Exe:



$f: \theta \longmapsto e^{2\pi i\theta}$

surjective, and is a Quotient map



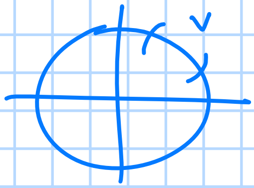
$f^{-1}(V) \Leftrightarrow V$
open open

claim: $\theta \longmapsto e^{2\pi i\theta}$ is a Quotient map from $[0,1]$ to S^1.

proof:

① surjective is trivial

② V is open in S^1



$V_{(a,b)} = \{e^{2\pi i\theta} \mid a < \theta < b\}$

here $a \geq 0$ $b < 1$
or $(a, b) = (1-\varepsilon, \varepsilon)$

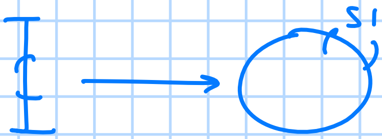
collection of open set which gives basis

now $\mathcal{B} = \{e^{2i\pi\theta} \mid a < \theta < b,\ a \geq 0, b \leq 1\} \cup \{e^{2\pi i\theta} \mid \theta \in (1-\varepsilon, 1] \cup [0, \varepsilon), \varepsilon > 0\}$

Basis of S

for $V \subseteq S^1$ so that $f^{-1}(V)$ open in $[0,1]$
$\Rightarrow V$ is open in S^1

case 1: $\theta\ (\neq 0, 1) \in f^{-1}(U)$

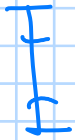


Case II : $0 \in V$

then $\theta = 0, 1 \in f^{-1}(V)$

now

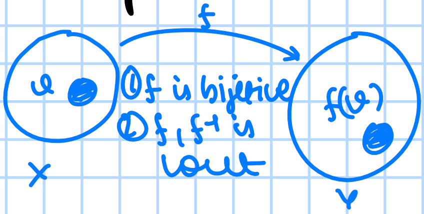
then $f^{-1}(V)$ to be open



this set has to be open $\Rightarrow f^{-1}(V)$ is open

14th Feb :

Recap : The notion of being the same space i.e via homeomorphism :



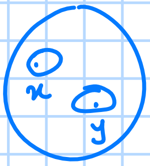
(Homeomorphism $\Leftrightarrow$ $\exists$ F s.t ① F is bijective ② F, F^{-1} is cont)

Given two spaces X, Y how do we check if they are homeomorphic ?
If they are, how do we compute homeomorphism ?

(these Questions are too complex to solve)

Next best thing to solve the above problems :
Study a few partitions of spaces which are invariant under homeomorphism

Property - I : Hausdorff



$\forall x, y$ s.t $x \neq y$
$\left.\begin{array}{l}\exists U_x \ni x \\ U_y \ni y\end{array}\right\}$ open s.t

$U_x \cap V_y = \phi$

(if X is Hausdorff, Y is not Hausdorff $\Rightarrow$ $X \not\cong Y$)

Exe : $\mathbb{R}$ is Hausdorff as $\forall x, y \in \mathbb{R}$

$\varepsilon = \dfrac{d(x,y)}{3}$

then $(x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \phi$

Similarly $\mathbb{R}^2$, $\mathbb{R}^d$ are Hausdorff

Note : (X, τ) with discrete topology i.e

$\tau = 2^X$ is always Hausdorff

Exe : Give an example of topology τ on $\mathbb{R}$, so that
① $(\mathbb{R}, \tau)$ is Hausdorff
② $\tau \neq$ standard or discrete (something in middle)

$\tau = \mathbb{R}$ std + $\{0\}$ being open

$\mathcal{B}_\tau = \{\text{Basis of } \mathbb{R}\} \cup \{\{0\}\}$

so if τ_1 is Hausdorff but τ_2 is not then
$\tau_1 \not\cong \tau_2$

Property - II : Connectedness

Def : we say a space X is not connected if $X = U \sqcup V$, where U, V are open in X
$\uparrow$ Disjoint
$U \neq X$
$V \neq X$

Exe : Take any space Z, for $U \in Z$, $V \in Z$ s.t
U, V are open
and $U \cap V = \phi$
then $X = U \cup V$ is a not connected space
$\downarrow$
subspace topology

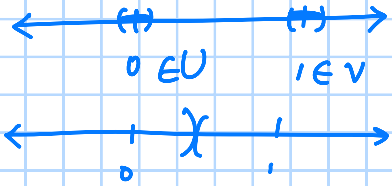
(this is not path connected)

Claim : $\mathbb{R}$ is connected

proof : Suppose $\mathbb{R}$ is not connected then
then $\exists U, V \subseteq \mathbb{R}$ s.t
$U \sqcup V = \mathbb{R}$

$U \cap V = \phi$
U, V open

wlog and $U \neq \phi$, $V \neq \phi$
now
wlog $0 \in U$
$1 \in V$
then



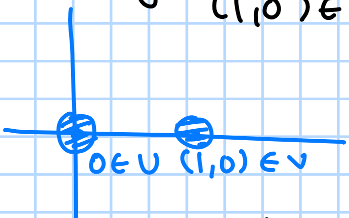
take $A = \{x \in U \mid 0 < x < 1\}$
then $\forall x \in A \Rightarrow x < 1$
so let $p_0 = \sup A$
least upper bound
then by costrution
$(p_0, 1] \subseteq V$
$(p_0 \notin U$ as U is open$)$

$\left(\begin{array}{l} A = \{x \in U \mid 0 < x < 1\} \\ \sup A \in V \\ \quad \text{as } A \text{ is open in } \mathbb{R} \\ \text{if } \sup A \in V \\ \quad \Rightarrow (\sup A - \varepsilon, \sup A + \varepsilon) \subseteq V \\ \quad \text{as } V \text{ is open} \\ \text{but } \sup A - \varepsilon/2 \in U, \text{ this is} \\ \text{a contradiction as } U \cap V = \phi \end{array} \right)$

now $p_0 \notin V$ as if $p_0 \in V$ then
$\exists \varepsilon > 0$ s.t
$B_\varepsilon(p_0) \subseteq V$
But p_0 is least upper boud
and $U \cap V = \phi$ $\divideontimes$

claim : $\mathbb{R}^2$ is connected

proof : suppose not, then $\mathbb{R}^2 = U \sqcup V$
wlog: $(0,0) \in U$
$(1,0) \in V$

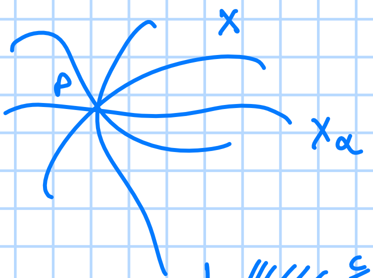


then we follow a similar proof to $\mathbb{R}$

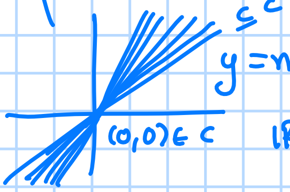
How do we prove that a space X is connected?

lemma : If $X_\alpha \subseteq X$, each X_α is connected and $\exists p \in \bigcap X_\alpha$
then $\bigcup X_\alpha$ is also connected

proof : suppose not then
$\bigcup_\alpha X_\alpha = C \sqcup D$
$C, D \neq \phi$ open in $\bigcup_\alpha X_\alpha$



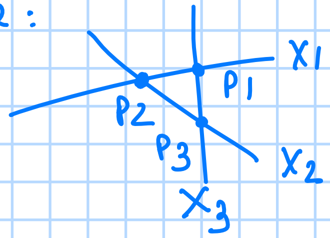
suppose $p \in C$ now,
open in X_1
$X_1 = (X_1 \cap C) \sqcup (X_1 \cap D)$ ← open in X_1
and as $p \in C \Rightarrow X_1 \subseteq C$
$\forall \alpha$, $X_\alpha \subseteq C$



$\mathbb{R}^2 = \bigcup_{m \in \mathbb{R}} \{x, mx\}_{\forall x \in \mathbb{R}} \Rightarrow \bigcup X_\alpha \subseteq C$ $\divideontimes$

suppose each X_α is connected and $X_\alpha \cap X_\beta \neq \phi$ $\forall \alpha \neq \beta$ then is $\cup X_\alpha$ connected?

Exe :



then if $P_1 \in C$
then
$X_1 \subseteq C$
$\Rightarrow X_2 \subseteq C$
and then $P_2 \in C$ and
$P_3 \in C$
$\Rightarrow X_3 \subseteq C$

So yes, even if by weakening condition

lemma: If $X_\alpha \subseteq X$, each X_α is connected and $\forall \alpha, \beta$
$X_\alpha \cap X_\beta \neq \phi$
then $\cup X_\alpha$ is connected

proof: as $\forall \alpha, \beta$ $X_\alpha \cap X_\beta \neq \phi$
for any two α, β
if $\cup X_\alpha$ is not connected
then
let $P \in X_\alpha \cap X_\beta$
then $\cup X_\alpha = C \cup D$
s.t $C \cap D = \phi$
$\cup X_\alpha \cap D \neq \phi, \cup X_\alpha \cap C \neq \phi, C \neq \phi, D \neq \phi$
wlog $P \in C \Rightarrow X_\alpha \subseteq C$ as X_α is connected
and $X_\beta \subseteq C$ as X_β is connected
then taking $X_{\alpha'}$ with X_α for all α'
$\Rightarrow X_{\alpha} \subseteq C$ $\forall \alpha'$
$\Rightarrow \cup X_\alpha \subseteq C$
$\Rightarrow \cup X_\alpha \cap D = \phi$ $*$
$\Rightarrow \cup X_\alpha$ is connected

(Here this is a more relaxed condition to show connectedness)

Equivalence condition of being connected

lemma: A space X is connected iff the only subsets A of X for which A is both closed and open is ϕ, X

proof : $(\Rightarrow)$ If X is not connected then
$X = U \sqcup V$
where U is open
V is open

(one more condition for connectedness)

$U^c = X - U = V \leftarrow$ open
so U is closed
$\Rightarrow U \neq X$
$\Rightarrow U$ is both open and close
not ϕ or X

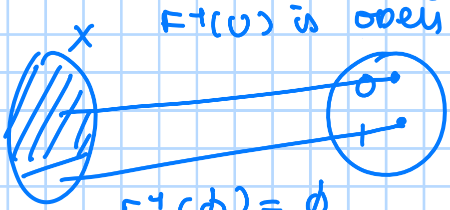
$(\Leftarrow)$ If U is s.t both open and closed
$U \neq \phi, X$
then
$X = U \cup U^c$
where $U \cap U^c = \phi$
$\therefore X$ is not connected

($U \sqcup V = X$ then
U, V open $\Rightarrow U = V^c \Rightarrow U$ closed)

Lemma: A space X is not connected $\Leftrightarrow \exists F : X \to \{0,1\}$ s.t $F(X) = \{0,1\}$
F is continuous (F is onto)

proof: $(\Rightarrow)$ $C = F^{-1}(0)$
$D = F^{-1}(1)$
$X = C \sqcup D$
and F is cont by construction

$F^{-1}(U)$ is open



$(\Leftarrow)$ let $C = F^{-1}(0)$
$D = F^{-1}(1)$
and $C \sqcup D = X$ is not connected

$F^{-1}(\phi) = \phi$
$F^{-1}(1) = U$
$F^{-1}(0) = V$
$F^{-1}(\{1,0\}) = X$

Exe: $X = \{A \subseteq M_2(\mathbb{R}) \mid AA^T = A^TA = I_2\}$ is connected ?

$X = U \sqcup V$ → open, → open

as det (A) for $A \in X$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$f : X \to \mathbb{R}$
$A \mapsto \det(A) = ad - bc$

now as $ad - bc$ is cont
$\Rightarrow f$ is cont

$\det(A) = \pm 1$

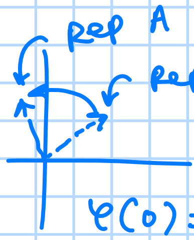
$f(X) = \{1, -1\}$ and f is cont then we apply the lemma
$\Rightarrow X$ is not connected

$X = \{A \mid \det A = 1\} \sqcup \{A \mid \det A = -1\}$

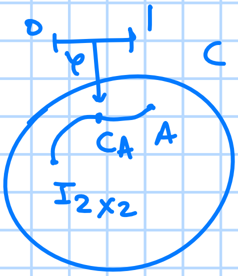
Exe: Is C above connected ?

$AA^T = A^TA = I_{2\times 2}$
$\det(A) = 1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$



$\varphi(0) = I_{2\times 2}$
$\varphi(1) = A$
is possible s.t $\varphi(t) \in C \;\; \forall t \in [0,1]$



$\varphi(0) = I$
$\varphi(1) = A$
then if φ is cont
$\Rightarrow I$ to A or C_A is connected
(path connected)

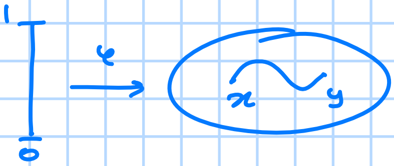
Def: (path connected) A space X is called path connected if given any two points $x, y \in X$, $\exists \gamma : [0,1] \to X$ s.t
$\gamma(0) = x$
$\gamma(1) = y$
γ has to be continious map

Note: If X is connected, Y is not connected $\Rightarrow X \not\cong Y$
$(\mathbb{R} \setminus \{0\} \not\cong \mathbb{R}^2 \setminus \{f(0)\})$

Note: C is path connected and so C is connected
(path connected $\Rightarrow$ connected)

18^{th} Feb :

Defn : A space X is called path connected if any two points in X can be "joined by a path". i.e given $x, y \in X$

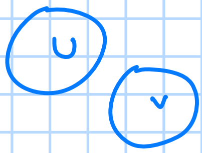


$\exists\ \varphi$ cont map s.t
$\varphi(0) = x$
$\varphi(1) = y$

$\left(\begin{array}{l} AA^T = I = A^TA \\ \det(A) = 1 \\ \text{is path connected} \end{array} \right)$

Lemma : X is path connected $\Rightarrow$ X is connected

proof : If $X = U \sqcup V$ i.e X is not connected



then $x_0 \in U$
$x_1 \in V$
if X is path connected then
$\exists\ \varphi : [0,1]$ s.t

$\varphi(0) = x_0$
$\varphi(1) = x_1$

(X is not connected $\Rightarrow$ X is not path connected)

$$\varphi[0,1] = \big(U \cap (\varphi[0,1])\big) \cup \big(V \cap (\varphi[0,1])\big)$$

there is a jump $\Rightarrow$ φ is not cont

$\therefore$ X is not connected $\Rightarrow$ X is not path connected

$\therefore$ X is path connected $\Rightarrow$ X is connected

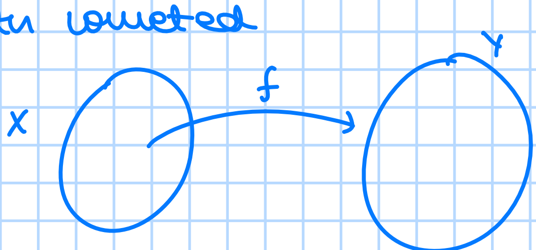
Exe : $[0,1]$, $(0,1)$, $[0,1)$, $\mathbb{R}$, $(0,\infty)$, $[0,\infty)$ are path connected

$\varphi(t) = (1-t)p + (t)q$
then for $p, q \in X$

$\varphi(0) = p$
$\varphi(1) = q$

property III : If X is not path connected but Y is not path connected then $X \not\cong Y$

Exe : X is path connected



path connected

if $f(X) = Y$ then $y_1, y_2 \in Y$ → For this f has to be onto

$y_1 = f(x_1)$
$y_2 = f(x_2)$ for $x_1, x_2 \in X$

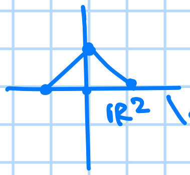
and $\varphi(t) = (1-t)x_1 + t x_2$

$\Rightarrow f(\varphi(t)) = \varphi'(t) = f((1-t)x_1 + t x_2)$

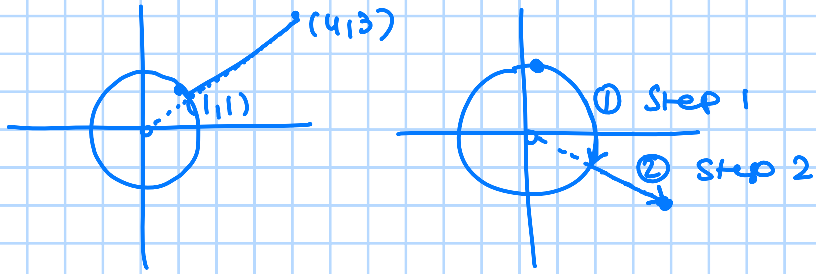
$f \circ \varphi$ is cont as ① f is cont
② φ is cont

Note: $f : X \longrightarrow Y$ is s.t
$f(X) = Y$, X is path-connected, f is cont
then Y is path connected

Exe:



$\mathbb{R}^2 \setminus \{0\}$ is path connected



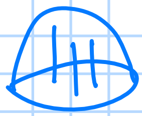
Exe: $S^1 = \{(x,y) \mid x^2+y^2=1\}$ is connected/path connected ?

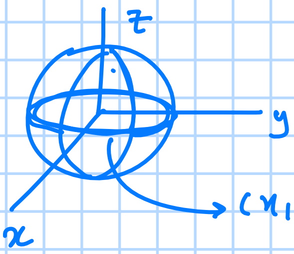
$[0,1] \xrightarrow{f} e^{2\pi i\theta} \in S^1$ — $[0,1]$ is path connected

then $f : [0,1] \to S^1$
is ① cont
② $[0,1]$ is path connected

$\Rightarrow S^1$ is path connected

Exe: $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

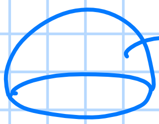




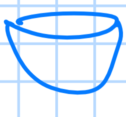
$\to (x,y,0)$ as $z=0$

$f : D^1 \longrightarrow \underbrace{S^2 \cap \{z \geq 0\}}_{\text{path connected}}$

D^1: path conn



$\to$ upper hemisphere $\cong D^2$ is path connected



$\cong D^2$

$\{(x,y) \mid x^2+y^2 \leq 1\} \xrightarrow{F} S^2 \cap \{z \geq 0\}$

$(x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$

$\Rightarrow S^2 \cap \{z \geq 0\}$ is path connected

Exe: $S^3 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2=1\}$

$S^2 \xrightarrow{f} S^3 \cap \{w \geq 0\}$
then

$S^3 \cap \{w \geq 0\}$ is path connected

$\Rightarrow S^3 \cap \{w \leq 0\}$ is path connected

lemma: If A is connected then $\bar{A}$ is also connected

proof:

If $\bar{A}$ is not connected then

$\bar{A} = U \sqcup V$

as $A \subseteq \bar{A}$ is connected

$\Rightarrow A \subseteq U$ (wlog)

as U is closed

$\Rightarrow \bar{A} \subseteq U$

$\Rightarrow V = \phi$ ✱

$\Rightarrow \bar{A}$ is connected

(A connected $\Rightarrow \bar{A}$ is connected)

$\bar{A} = U \sqcup V$
$A \subseteq \bar{A}$ is connected
$A \subseteq U$ or $A \subseteq V$
say $A \subseteq U$
then as U is closed
$\Rightarrow \bar{A} \subseteq U$
$\Rightarrow \bar{A} \subseteq U$
$\Rightarrow V = \phi$ ✱

Exe: show that ∃ ? connected but not path connected

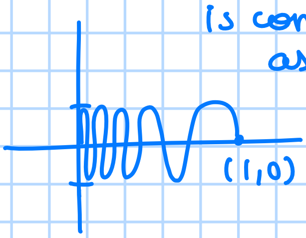
$X = \{(t, \sin\frac{2\pi}{t}) ; 0 < t \leq 1\}$

is connected

as $t \longrightarrow \sin\frac{2\pi}{t} \to$ cont

↓

graph



now let $\bar{X} = \{(t, \sin\frac{2\pi}{t}) ; 0 < t \leq 1\} \cup \{(0,x) ; -1 \leq x \leq 1\}$

as X is connected

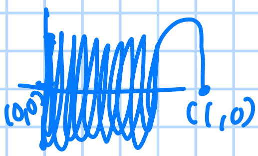
$\Rightarrow \bar{X}$ is connected

(if $\bar{X}$ is path connected then cont.)

now if $\exists\, \varphi : [0,1] \longrightarrow \bar{X}$
s.t

$\varphi(0) = (1,0)$

$\varphi(1) = (0,0)$



let $t_0 = \max\{t < 1 \mid \varphi[0,t] \subseteq X\}$

now $\varphi[0, t_0]$

$\varphi(t_0) = (0, \beta)$ for $\beta \leq 1$

wma $\beta \leq 0$ (wlog)

$\varphi([0, t_0)) \subseteq X$

$\varphi(t) = (x(t), \sin\frac{2\pi}{x(t)})$

$0 \leq t < t_0$

path

$x(t)$: $x(t_0)=0$
$x(t)>0 \quad t<t_0$
$x(t)$ is cont
if $t_n \longrightarrow t_0$
then $x(t_n) \longrightarrow 0$

now $[t_0-\frac{1}{n}, t_0)$

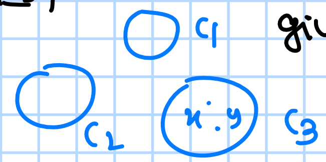
$$\left\{ \frac{1}{x(t)} ; \quad t_0-\frac{1}{n} < t < t_0 \right\}$$

$\supseteq [c_n, \infty)$
as $x(t)$ is cont and t is connected
$\frac{1}{x(t)}$ is cont and connected
now $\exists \, t_n \longrightarrow t_0$ s.t

$$\left(x(t_n), \sin\left(\frac{2\pi}{x(t_n)}\right) \right) = \left(x(t_n), 1\right) \xrightarrow[\text{as } n\to\infty]{} (0,1)$$

where φ is cont $\Rightarrow \varphi(t_0)=(0,1) \neq (0,\beta)$ $*$
as $\beta \leq 0$ (wlog)

Is it possible to break a space into disjoint connected spaces?
Yes,



given $x, y \in X$
define $x \sim y$ if $\exists \, C \subseteq X$
C is connected, $x, y \in C$

this is an equivalence relation

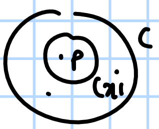
let $\{C_{x_i}\}$ be set of equivalence classes

each claim: ① each C_{x_i} is connected
② each C_{x_i} is closed
③ if $p_0 \in C_{x_i}$ then C_{x_i} is largest connected set cont p_0
④ $X = \bigsqcup C_{x_i}$

① each C_{x_i} is connected as:
$x \sim y$ if $\exists \, C \subseteq X$, C is connected
then C_{x_i} by defination is connected

② each C_{x_i} is closed:
let $x \in \overline{C}$
but $x \notin C$
then
as $x \in \overline{C}$, $\exists$ nbd of x say U s.t
$C \cap U \neq \phi \Rightarrow U \subseteq C$
$\Rightarrow x \in C$ $*$
so $\overline{C} = C \Rightarrow C$ is closed

③ if $p_0 \in C_{x_i}$ and C_{x_i} is not largest
say $C_{x_i} \subseteq C$ for $y \in C \setminus C_{x_i}$
as $p_0 \in C$ and $y \in C$
$\Rightarrow p_0 \sim y$
$\Rightarrow p_0 \in C_{x_i}$ and $y \in C_{x_i}$
$\Rightarrow$ this is a contradiction



④ $\bigsqcup C_{x_i} \subseteq X$

now $\forall x \in X \Rightarrow x \in$ one of the conned set say C_{x_i}

$\Rightarrow x \in \bigsqcup C_{x_i}$

$\Rightarrow X \subseteq \bigsqcup C_{x_i}$

so $X = \bigsqcup C_{x_i}$

Defn: Each C_{x_i} is called connected component of X

Exe: compute C_{x_i} of $X = \mathbb{R}$

$X = \mathbb{R}$ → largest connected set $= C_{x_1} = \mathbb{R}$

Exe: $X = (0,1) \cup (2,3)$

as $\frac{1}{2} \in (0,1)$ connected

$\Rightarrow C_{x_1} = (0,1)$

$C_{x_2} = (2,3)$

4th March :

Property 4: Compactness : (one more property to check homeomorphism)

Defn : A metric space (X,d) is called compact if for any sequence $x_n \in X$ it converges to a point in X along a subsequence.

Theorem: A metric space (X,d) is compact iff every open cover of X has a finite subcover.
(given any collection of open sets $\exists$ a finite subcover, covering X)

finite covering of X

$X = \bigcup_{\alpha} V_\alpha$, $\exists$ finite $V_{\alpha_1}, V_{\alpha_2}, \ldots V_{\alpha_N}$ s.t $X = \bigcup_{i=1}^{N} V_{\alpha_i}$

proof: ($\Leftarrow$) suppose every open cover has a finite subcover, then suppose $\{x_n\}$ sequence in X

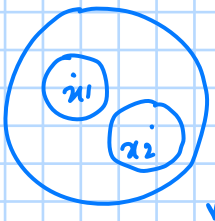
if x_n does not converge along any subsequence

$x_0, x_1, x_2, \ldots\ldots \leftarrow$ does not converge
since it does not converge along subsequence,
$x_1 \rightarrow$ not a limit point
as for x_1 to be limit point
$\forall$ nbd of x_1
$\exists$ some $x_i \in$ nbd of x_1
$\downarrow$
infinite many
$\Rightarrow \exists$ subseq conv to x_1 $\divideontimes$

so x_1 is not a limit point

$\therefore \exists\, r_1 > 0$ s.t
$B(x_1, r_1) \cap \{x_2, x_3, \ldots\} = \phi$

this holds for all x_i
$\Rightarrow A = \{x_1, x_2 \ldots\}$ has no limit point



A has no limit point
$\Rightarrow$ A is closed set $(\because \bar{A} = A)$

so $X = \overset{\text{open}}{A^c} \cup \overset{\text{open}}{\bigcup_{i=1}^{\infty} B(x_i, r_i)}$ (cover of X)

$\Rightarrow$ as X is compact $X = A^c \cup \bigcup_{j=1}^{N} B(x_{k_j}, r_{k_j})$ $\divideontimes$

($\Rightarrow$) suppose every sequence in X has a convergent subsequence.
this tells us that $\forall \varepsilon > 0$ (fixed)
it is not possible to find infinitely balls mutually disjoint balls in X, each of them having radius ε. } Not totally bond
(if do then $\exists$ a sequence of centre s.t it has a subsequence converging)

we want to show if $X = \bigcup_{\alpha \in I} V_\alpha$ then $\exists$ only finitely many $V_{\alpha_1}, V_{\alpha_2} \dots V_{\alpha_N}$ $X = \bigcup_{i=1}^{N} V_{\alpha_i}$

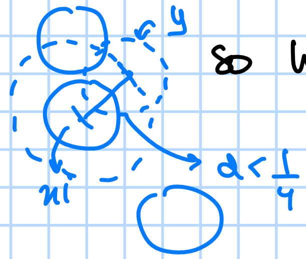
now special case

$V_\alpha = B(x_\alpha, 1)$

suppose it is not possible to cover X by finitely many of these balls.

for $\varepsilon = \frac{1}{8}$ we have finite disjoint balls say 10 s.t $\forall y \in X$

$d(y, x_i) < \frac{1}{4}$ for some x_i from centre of 10 balls.

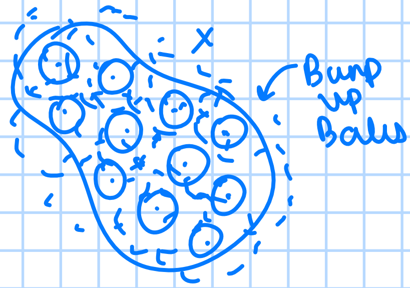


so here $x_1, x_2 \dots x_{10}$ s.t

$\bigcup B_{\frac{1}{4}}(x_i)$ covers X

or $\bigcup B_1(x_i)$ cover X

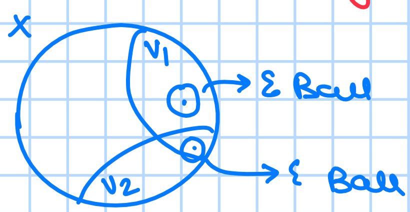
so finitely many



now if V_α is a general open set then we can use lemma proved below that $\exists \varepsilon > 0$ s.t $\forall x \in X$, $\exists \alpha \in I$ s.t $B(x, \varepsilon) \subseteq V_\alpha$

and then this is now our special case.

lemma : $\exists \varepsilon > 0$ s.t given $x \in X$, $\exists \alpha \in I$ s.t $B(x, \varepsilon) \subseteq V_\alpha$



7th march:

Theorem: let X be a metric space then the following are equivalent:

① X is compact i.e every sequence in X has a conv subsequence
② Every open cover has a finite subcover.

proof: ($\Rightarrow$) let

$X = \bigcup V_\alpha$

want finitely many $V_{\alpha i}$ so that

$$X = \bigcup_{i=1}^{N} V_{\alpha i}$$

Special case was when $V_\alpha = B(x_\alpha, 1)$
then this is finite (proved above)

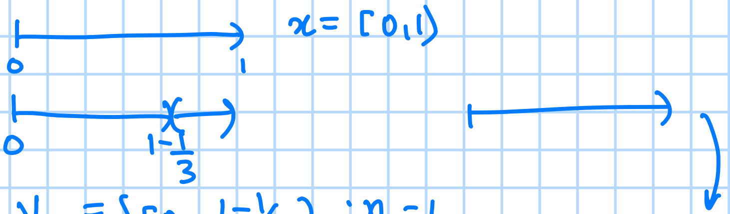
so now we wanted to prove this lemma, use it and so we are done. use the below lemma.

$$\text{so, } X = \bigcup_{x \in X} B(x, \varepsilon_0) \Rightarrow X = \bigcup_{i=1}^{N} B(x_i, \varepsilon_0) = \bigcup_{i=1}^{N} V_{\alpha_i}$$

from lemma as $B(x_i, \varepsilon_0) \subseteq V_{\alpha_i}$

singular by lemma

lemma: $\exists \varepsilon_0 > 0$ s.t $\forall x \in X, \exists \alpha = \alpha(x)$ s.t $B(x, \varepsilon_0) \subseteq V_\alpha$ (X is compact)



$$V_n = \begin{cases} [0, 1-\frac{1}{3}) & ; n = 1 \\ \left(1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}\right) & ; n \geq 2 \end{cases}$$

$|V_n| \sim \frac{1}{n^2}$

This fails here (cause X is not compact) as it does not have a finite subcover but we will prove

proof: suppose not, then $\exists x_i \in X$ s.t $B(x_i, \varepsilon) \nsubseteq V_\alpha$ for any ε, $\forall \alpha \in I$ (find ε, x_i)

let $\varepsilon = 1$ (wlog case)

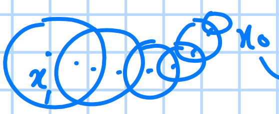
$B(x_1, 1) \nsubseteq V_\alpha \ \forall \alpha$

$\exists x_2 \in X$ s.t
$B(x_2, \frac{1}{2}) \nsubseteq V_\alpha \ \forall \alpha$

$\exists x_3 \in X$ s.t
$B(x_3, \frac{1}{3}) \nsubseteq V_\alpha \ \forall \alpha$

$\vdots$

But $x_n \longrightarrow x_0$ along a subsequence



limit point

V_{α_0} $\to B(x_{2N}, \frac{1}{2N})$

↙ After a point $B(x_0, x_N) \subseteq V_{\alpha_0}$

let $B(x_0, \frac{1}{N}) \subseteq V_{\alpha_0}$ (let $x_0 \in X$, $\exists\, V_{\alpha_0} \ni x_0$ as V_{α_0} is open, $\exists N$)

($\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $|x_{N_0} - x_0| < \varepsilon$)

$\forall \varepsilon > 0; \varepsilon = \frac{1}{2N}$ $|x_k - x_0| < \frac{1}{2N}$ (as $x_n \to x_0$)

$\exists K \in \mathbb{N}$ $\quad K_0 = \max\{K, 2N\}$

$\Rightarrow B(x_{K_0}, \frac{1}{K_0}) \subseteq B(x_0, \frac{1}{N}) \subseteq V_{\alpha_0}$

we

so $B(x_{2N}, \frac{1}{2N}) \not\subseteq V_\alpha \;\forall \alpha$

this is a contradiction.

and $|x_{2N} - x_0| < \frac{1}{2N}$

Defn: A space X is called compact if every open cover has a finite subcover.
(general X, X can/cannot be metric space)

lemma: $[0,1]$ (subspace topology) is compact

proof: to show that $\forall x_n \in [0,1]$ has a conv subsequence

(Remove Repetations)

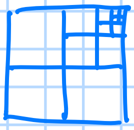
$|[0, \frac{1}{2}) \cap \{x_n : n \geq 1\}| = \infty$ wlog case

$\Rightarrow |x_m - x_n| \leq \frac{1}{2^i} \;\forall n \neq m$

(we use X is metric space compact if every seq has sub seq)

we repeate this to get the diameter as $\frac{1}{2^i} \to 0$ so every x_n has conv subseq

Exe: $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ is compact, as



$x_n \to x$
$y_n \to y$ (first method)

or $(x_n, y_n) \to (x, y)$
iff $x_n \to x$ in $[0,1]$ (second method)
$y_n \to y$ in $[0,1]$

so for $[0,1]^d$ same.

Non-example: $(0,1)$

so by defination, every open cover has finite cover

if

$V_n = (\frac{1}{n}, 1)$

then $\bigcup_{n=1} V_n = (0,1)$ but not finitely many

So if A is compact, and x_0 is a limit point of A, then $A \setminus \{x_0\}$ is not compact

let $V_n = (B(x_0, \frac{1}{n}))^c \cap A \setminus \{x_0\}$

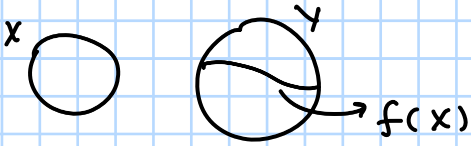
open cover

union covers everything but need infinite many

(A is compact x_0 limit point
x_0 is a limit point but not in A then we can construct this)

lemma: If X is compact and $f: X \to Y$ is continious then $f(Y)$ is compact

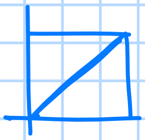
proof:



(f is cont iff $f^{-1}(V)$ open $\forall V$ open in Y)

for any infinite cover of Y, then inverse of all open sets of cover $X \Rightarrow \exists$ finite cover of X $(f^{-1}(U))$
$\Rightarrow \exists$ finite cover of Y (U)

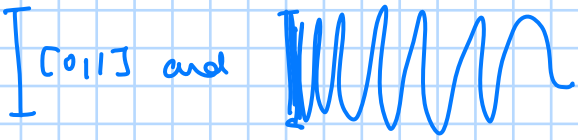
Exe:



$Y = \{(x,x) \,;\, 0 \le x \le 1\}$
then let
$f(x) = (x,x)$
then f is continious
$\Rightarrow Y$ is compact

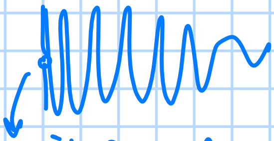
Note: If $X \cong Y$ then:
X is compact $\Leftrightarrow Y$ is compact
but X compact, Y compact $\not\Rightarrow X \cong Y$
as $[0,1] \ncong [0,1]^2$

Exe:



$[0,1]$ and are both compact
but
not homorphic
as $[0,1] \setminus \{p\}$ is not path connected

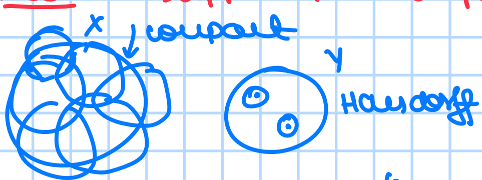
but



this point removed is still connected

$A \subseteq B \subseteq \bar{A} \rightarrow$ connected
$\downarrow$ connected $\rightarrow$ conneted (some limit points)

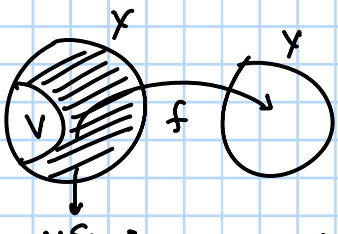
Theorm: suppose X is compact and Y is housdorff, then for any



$f : X \longrightarrow Y$
f is bijetice and cont
then
f is homeomorphism

as $[0,1] \ncong [0,1]^2$ but $X = [0,1]$ is compact
$Y = [0,1]^2$ is hausdorff
so $f : X \longrightarrow Y$ if bijective and cont
then $X \cong Y$
$\therefore$ no such f exist s.t $f : X \longrightarrow Y$ is bijetive and continious. (corr of theorm)

proof:



V^c is compact

so we need to show that $f(V)$ open in Y $\forall$ V open in X.

(V is open in $X \Rightarrow f(V)$ open in Y)

we have to first show that V^c is compact suppose $V^c = \bigcup (W_\alpha \cap V^c)$

W_α are open in X

open cover of V^c

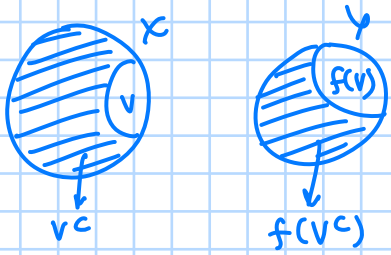
$X = \bigcup W_\alpha \cup V$ → this is an open cover for X

$\Rightarrow X = \bigcup_{i=1}^{N} W_{\alpha_i} \cup V \Rightarrow V^c \subseteq W_1 \cup W_2 \cdots \cup V$

$\Rightarrow V^c = \bigcup_{i=1}^{N} (W_{\alpha_i} \cap V^c)$

so V^c is compact

(f is cont $\Rightarrow f(V^c)$ is compact)



$f(V^c)$ is compact
now if we show that $f(V^c)$ is closed subset of Y then we are done.

so to show that $\forall y \in f(V)$
y cannot be a limit point of $f(V^c)$

(this means if x_0 is a limit point of $f(V^c)$ then $x_0 \in f(V^c)$)

now $\forall x \in f(V^c)$ as $x \neq y$
$\exists U_x$ → open in Y
U_y → open in Y

then as $f(V^c)$ is covd by $\{U_x\} \forall x \in f(V^c)$

then $\exists \{U_{x_1}, U_{x_2}, \ldots U_{x_n}\}$ covg $f(V^c)$
as $f(V^c)$ is compact

so $U_{y_i} \cap U_{x_i} = \phi \quad \forall i = 1, 2 \ldots n$

then $U = U_{y_1} \cap U_{y_2} \ldots \cap U_{y_n}$ ($y \in U_{y_i} \; \forall i$ by defination)

open — finite intersection of open sets

$y \in U$ s.t

$U \subseteq f(V)$ ($\because U \cap U_{x_i} = \phi \; \forall i$
$\Rightarrow U \cap f(V^c) = \phi$)

$\Rightarrow y$ is not a limit point

$\therefore \forall y \in f(V)$ it is not a limit point

$\Rightarrow f(V^c)$ is closed

$\Rightarrow f(V)$ is open

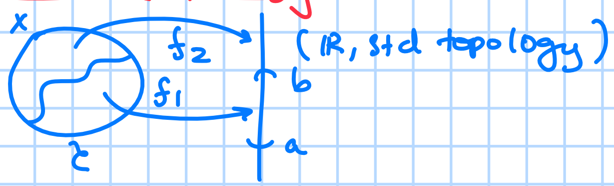
(this proof can also be seen as $\forall y \in f(V)$, $\exists U = U_{y_1} \cap U_{y_2} \ldots \cap U_{y_n}$ s.t $U \subseteq f(V)$
$\Rightarrow f(V)$ is open)

12th march:

Theorm: (Tychnoff theorm) suppose X_α, $\forall \alpha \in I$ are compact, then so is $\prod_{\alpha \in I} X_\alpha$ (product topology)

Exe: $[0,1]$ is compact $\Rightarrow [0,1]^n$ by Tychnoff theorm

Product topology:



A basis for the smallest topology on X which makes both f_1, f_2 continious is given by:

(subbasis) $\mathcal{C}_\tau = \{ f_1^{-1}(a,b) \mid a<b \} \cup \{ f_2^{-1}(a,b) \mid a<b \}$

$$\mathcal{B}_\tau = \{ f_1^{-1}(a,b) \cap f_2^{-1}(c,d) \mid a<b,\ c<d \}$$

So an example of an open set in X is $f_1^{-1}(0,1) \cap f_2^{-1}(3,4)$

Same thing works for any $f_\alpha : \underset{\text{set}}{X} \to (Y, \text{space})$

meaning the smallest topology on X which makes each of the $f_\alpha : X \to Y$ continious has a natural basis given by

$$\mathcal{B}_\tau = \{ \bigcap_{i=1}^{N} f_{\alpha_i}^{-1}(V_i) \mid V_i \text{ is open in } Y,\ N \geq 1 \}$$

Defn: suppose X_α are spaces, then consider the set

$$\prod_{\alpha \in I} X_\alpha = \{ (x_\alpha)_{\alpha \in I} \}$$

and consider projection map

$$\pi_{\alpha_0} : \prod_{\alpha \in I} X_\alpha \to X_{\alpha_0}$$

$$\pi_{\alpha_0}((x_\alpha)_{\alpha \in I}) = x_{\alpha_0}$$

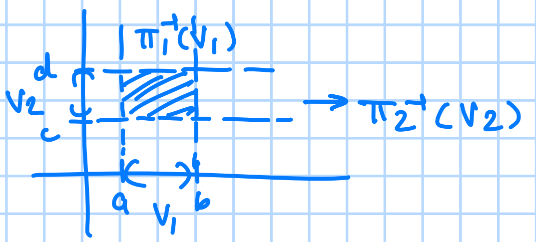
$$\prod X_\alpha \xrightarrow{\pi_{\alpha_0}} x_{\alpha_0} \in X_{\alpha_0}$$

$(x_\alpha)_{\alpha \in I}$

The smallest topology on X which makes each π_α continious is called the product topology

Exe: $X_1 = X_2 = \mathbb{R}$

$X_1 \times X_2 = \mathbb{R}^2 = \{ (x,y) \mid x,y \in \mathbb{R} \}$



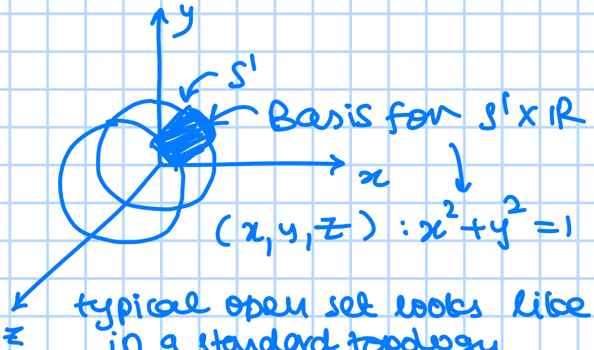
A Basis for the $\mathbb{R}^2$ product topology is

$$\pi_1^{-1}(a,b) \cap \pi_2^{-1}(c,d) = \{ (x,y) \mid a<x<b,\ c<y<d \}$$

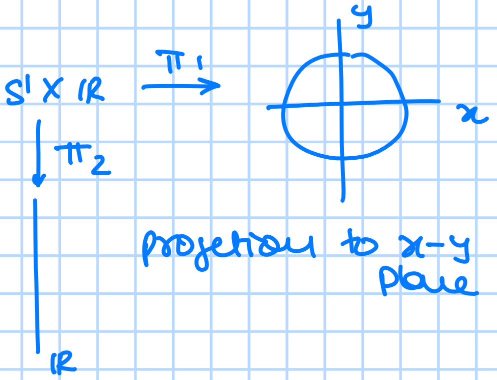
$$= \underbrace{(a,b)}_{\text{open set in } \mathbb{R}} \times \underbrace{(c,d)}_{\text{open set in } \mathbb{R}}$$

Exe: $X_1 = S^1 \quad X_2 = \mathbb{R}$

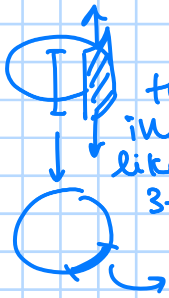
$X_1 \times X_2 = S^1 \times \mathbb{R}$ as a set



typical open set looks like that in a standard topology

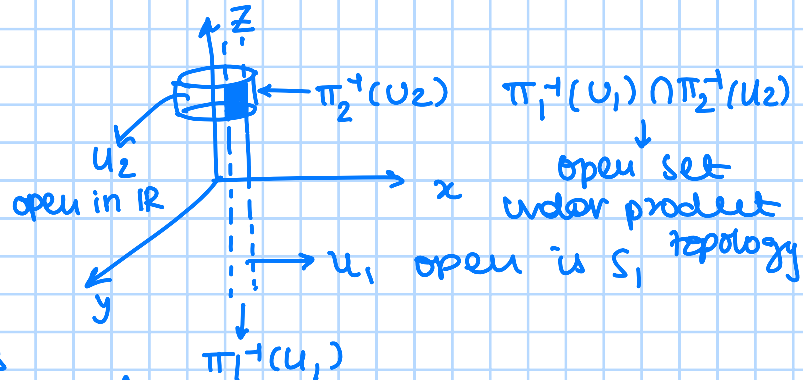


projection to x-y and $\mathbb{R}$ plane



this projection inverse would look like a disc corresponding 3-axis coordinates

projection inverse looks like infinite curved strip



$\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2)$ open set under product topology

now back to tychonoff theorm, we will first prove the special case when

$$X = X_1 \times X_2$$

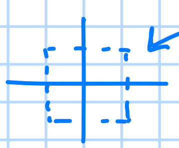
and every X_i is a metric space

lemma: There is a metric d on $X_1 \times X_2$ s.t $(X_1 \times X_2, \text{metric topology}) = (X_1 \times X_2, \text{product topology})$

proof: Define d on $X_1 \times X_2$ as:

$$d((\underset{\in X}{x_1}, x_2), (\underset{\in X}{y_1}, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$$

(if $X_1 = X_2 = \mathbb{R}$ $\quad B_d((0,0),1) = \{(x,y) \mid d((x,y),(0,0)) < 1\}$



$$= \{(x,y) \mid \max(|x|, |y|) < 1\}$$

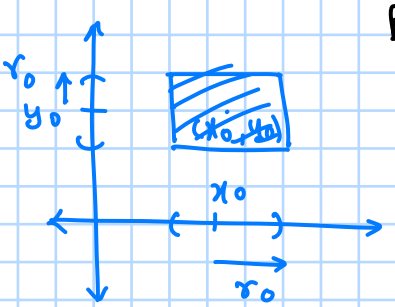
$$= \{(x,y) \mid |x| < 1, |y| < 1\} = (-1,1) \times (-1,1))$$

we claim that this d gives us product topology on $X_1 \times X_2$

so we have to show any ball $B_d((x_0,y_0),r_0)$ is open in the product topology (the ball is ofc open in metric topology)

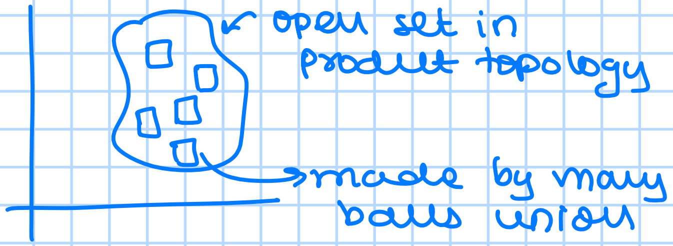
$$B_d((x_0,y_0),r_0) = \{(x,y) \mid d_1(x,x_0) < r_0 \wedge d_2(y,y_0) < r_0\}$$

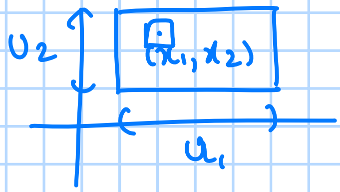
$$= \pi_1^{-1}(B_{d_1}(x_0,r_0)) \cap \pi_2^{-1}(B_{d_2}(y_0,r_0))$$



(Ball in metric topology = product of two balls in product topology)

now, for any U open in product topology then it is open in metric topology





$\exists r_1, r_2$ s.t $B_{d_1}(x_1, r_1) \subseteq U_1$
$B_{d_2}(x_2, r_2) \subseteq U_2$
$r = \min(r_1, r_2)$
then
$$\underbrace{B_d((x_1, x_2), r)}_{\text{open in } X_1 \times X_2} \subseteq U_1 \times U_2$$

so we have
$$\underbrace{\mathcal{B}_{X_1 \times X_2}}_{\text{Basis of product topology}} = \{ B_d((x_1, x_2), r) \mid d > 0,\ x_1 \in X_1,\ x_2 \in X_2,\ r > 0 \}$$

and now $\mathcal{B}_{(X_1 \times X_2, d)} = \{\text{Balls wrt } d \text{ by defination}\}$

$\Rightarrow \mathcal{B}_{(X_1 \times X_2, d)} = \mathcal{B}_{X_1 \times X_2}$

$\Rightarrow$ product topology = metric topology wrt d

Theorm : (Tychnoff special case) suppose X_1, X_2 are metric spaces and also compact then so is $X_1 \times X_2$ (product topology)

proof : let $(x_1, y_1), (x_2, y_2), \ldots$ be sequence in $X_1 \times X_2$

now as X_1 is compact, $x_n \to x_0$ along a subsequence
let $\{x_{n_i}\}_{i \in \mathbb{N}}$ be s.t
$$x_{n_1}, x_{n_2}, \ldots \to x_0$$
now choose this n_i's and check y_n on these indexes
i.e consider the subsequence :
$$y_{n_1}, y_{n_2}, \ldots$$
as X_2 is compact
$y_{n_i} \to y_0$ along a subsequence
say
$$y_{n_{i_1}}, y_{n_{i_2}}, \ldots \to y_0$$
now take n_{i_j}'s index from both x_i's and y_i's
then
$$(x_{n_{i_j}}, y_{n_{i_j}}) \to (x_0, y_0)$$
or for $\{(x_i, y_i)\}_{i \in \mathbb{N}}$ it also converges by a subsequence. As this is general for any $\{(x_i, y_i)\}_{i \in \mathbb{N}}$

$\Rightarrow$ every sequence in $X_1 \times X_2$ converges along a subsequence

$\Rightarrow X_1 \times X_2$ is compact

18th march:

Quiz-3: Next Thursday 27th march, syllabus after midsem to friday (this week)

Theorem: (Tychonoff theorem) product of compact spaces is compact under the product topology

Note: We proved the special case when X, Y are compact metric spaces then $X \times Y$ is also compact with natural metric

$$d\big(\underset{\in X \times Y}{(x,y)}, \underset{\in X\ \in Y}{(z,w)}\big) = \max\big(d_X(\underset{\in X}{x}, \underset{\in X}{z}), d_Y(\underset{\in Y}{y}, \underset{\in Y}{w})\big)$$

$X \times Y$ as metric topology w.r.t. d $= X \times Y$ product topology

Note: $A \subseteq X$, a metric space with metric d, then

$$\bar{d}(a,b) = d(a,b)$$

is also a metric on A

so, subset of metric space is also a metric space

Eg: $S^1 \to$ sits inside $[-2,2]^2$

closed and bounded and $[-2,2]$, $[-2,2]$ is compact metric

$\Rightarrow [-2,2]^2$ is compact (first case of Tychnoff)

$S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$

$= f^{-1}(0)$ where $f(x,y) = x^2 + y^2 - 1$ — continuous

$\bar{d}((a,b),(c,d)) = \sqrt{(a-c)^2 + (b-d)^2}$

$\Rightarrow f^{-1}(\text{closed})$ ($\because \{0\}$ closed $\Rightarrow f^{-1}(0)$ is closed)

$\Rightarrow f^{-1}(0) = S^1$ is closed

$\Rightarrow S^1 \subseteq [-2,2]^2$ & closed

$\Rightarrow S^1$ is a compact metric space

so now $S^1 \times S^1 \Rightarrow$ is also compact by special case of tychnoff

so is $S^1 \times S^1 \times S^1$ and so on

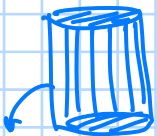
$\therefore$ by induction

for $n \in \mathbb{N}$

$$\underbrace{S^1 \times S^1 \times \cdots S^1}_{n \text{ times}} = T^n$$

n dimensional torus

Eg: $S^1 \times [0,1]$ is also a compact metric space

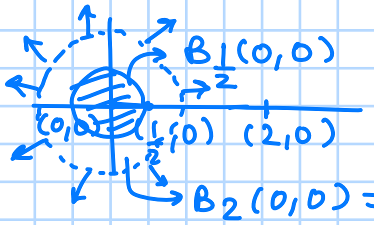


$\subseteq \mathbb{R}^3$

closed cylinder

lemma: different metrics can lead to the same topology

Eg: $\bar{d}((x_1,y_1),(x_2,y_2)) = \begin{cases} (|x_1 - x_2|^2 + |y_1 - y_2|^2)^{\frac{1}{2}} & ; L < 1 \\ 1 & ; L \geq 1 \end{cases}$



lemma : $(\mathbb{R}^2, \text{std topology}) = (\mathbb{R}^2, \bar{d})$

proof :

Basis $\mathcal{B}$ ← this is what topology depends on

$$\mathcal{B}_{\mathbb{R}^2} = \{ B_d((x,y), r) \mid (x,y) \in \mathbb{R}^2,\ r>0, r<1 \}$$

when $r<1$ we have

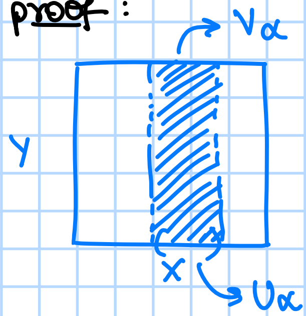
$$B_{\bar{d}}((x,y),r) = B_d((x,y),r)$$

so

$$\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{(\mathbb{R}^2, \bar{d})} \Rightarrow (\mathbb{R}^2, \text{std topology}) = (\mathbb{R}^2, \bar{d})$$

Theorem: If X, Y are compact then so is $X \times Y$ with product topology

proof :



suppose $\{V_\alpha\}$ covers $X \times Y$ i.e

$$X \times Y = \bigcup_{\alpha \in I} V_\alpha$$

now in a special case of $\{V_\alpha\}$

let

$$V_\alpha = \pi_1^{-1}(U_\alpha) = U_\alpha \times Y$$

U_α is open in X

look at the collection $U_\alpha, \alpha \in I$

$$X = \bigcup_{\alpha \in I} U_\alpha \text{ and } X \text{ is compact}$$

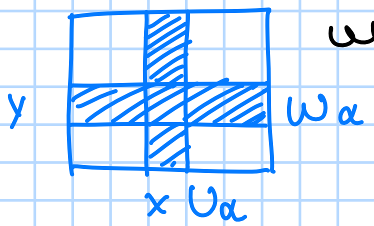
$\Rightarrow$ we need finite U_α to n cover X

$\Rightarrow X = \bigcup_{i=1}^{n} U_{\alpha_i}$ for some $n \in \mathbb{N}$

now $X \times Y = \bigcup_{i=1}^{n} (U_{\alpha_i} \times Y)$

$$X \times Y = \bigcup_{i=1}^{n} V_{\alpha_i} \text{ ← finite subcover}$$

proof for general case, we have V_α st they are general (not open strips)



wlog assume each $V_\alpha = U_\alpha \times W_\alpha$

U_α → open in X, W_α → open in Y

as if not then we can decompose it to this as

$$\mathcal{B}_{X \times Y} = \{ U \times V \mid U \in \tau_X, V \in \tau_Y \}$$

so $V_\alpha = \bigcup_{\alpha \in I} B_\alpha \quad B \in \mathcal{B}_{X \times Y}$

we will assume each

$V_\alpha = U_\alpha \times W_\alpha$ (Basis elem of $X \times Y$)

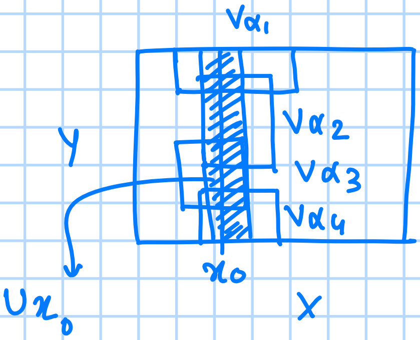
let's fix $x_0 \in X$, look at the set $x_0 \times Y$

as $Y \cong x_0 \times Y$ (trivial)

$\Rightarrow x_0 \times Y$ is compact

so $\exists\, V\alpha_i$, $i=1,2\ldots k$ s.t

$x_0 \times Y$ is covered by $\{V\alpha_i\}$



now, this is s.t

$$V\alpha_1 = U\alpha_1 \times W\alpha_1$$
$$\vdots$$
$$V\alpha_k = U\alpha_k \times W\alpha_k$$

now let

$$U_{x_0} = U\alpha_1 \cap U\alpha_2 \ldots \cap U\alpha_k$$

then $U_{x_0} \times Y$ will be covered by finitely many $V\alpha$

now $\forall x \in X$, $U_{x_0} \times Y$ will be open and will be covered by finite many $V\alpha$

$\Rightarrow \exists$ subcollection of $\{U_{x_{0_1}} \times Y, U_{x_{0_2}} \times Y \ldots U_{x_{0_m}} \times Y\}$ to cover X

$(\because X = U_{x_{0_1}} \cup U_{x_{0_2}} \cdots U_{x_{0_m}})$

$\Rightarrow$ we need finite many $V\alpha$ to cover $X \times Y$

$$\left(\because X \times Y = \bigcup_{i=1}^{m} (U_{x_{0_i}} \times Y)\right)$$

lemma: Suppose V is open in $X \times Y$. Then $\exists$ open sets of the form $U_i \times W_i$ s.t

$$V = \bigcup_{i \in J} U_i \times W_i$$

U_i is open in X, W_i is open in Y

proof: $\mathcal{B}_{X\times Y} = \{U_i \times W_i \mid U_i \text{ is open in } X, W_i \text{ is open in } Y\}$

then

$$V = \bigcup_{\alpha \in I} B_\alpha = \bigcup_{\alpha \in I} U_\alpha \times W_\alpha$$

lemma: By induction on N, $X_1 \times X_2 \ldots \times X_N$ is compact if X_i is compact $\forall i = 1, 2, \ldots N$

proof:

Trivial proof, followed using Induction

Theorem: If each $X_1, X_2 \ldots\ldots$ is compact then so is $X_1 \times X_2 \times \cdots\cdots$

Eg: let's try to see what a typical open set in $\mathbb{R} \times \mathbb{R} \times \ldots\ldots$ look like.

$$\pi_1^{-1}(0,1) \text{ is open in } X = \{(x_1, x_2, \ldots\ldots) \mid 0 < x_1 < 1\}$$
$$= (0,1) \times \mathbb{R} \times \mathbb{R} \cdots\cdots$$

A typical open set can be like:

$$\underbrace{(0,1) \times (0,2) \times \ldots\ldots}_{\text{only finite many like this}} \times \mathbb{R} \times \mathbb{R} \times \cdots$$

as $(0,1) \times (0,1) \times \cdots\cdots$ in $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots\cdots$ is not open as basis of product topology is given by $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cdots\cdot \cap \pi_n^{-1}(U_n)$ ← finite many

where $U_1, U_2 \ldots U_n$ are open in $\mathbb{R}$ and $n \geq 1$

$$\mathcal{B} = \{\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cdots\cdot \pi_n^{-1}(U_n) \mid U_i \text{ open in } \mathbb{R}, n \geq 1\}$$

$\mathcal{B} = \{ U_1 \times U_2 \times \cdots U_n \mid U_1, U_2 \ldots U_n$ open in $\mathbb{R}$ and n can be any number but has to be fixed $\}$

for $(0,1) \times (0,1) \times \cdots\cdots = A$

$\left(\frac{1}{2}, \frac{1}{2}, \ldots\ldots\right) \in A$, if A is open then $\exists$ nbd U of $\left(\frac{1}{2}, \ldots\right)$ s.t $U \subset A$

where U is open

$U = \underbrace{U_1 \cap U_2 \cap U_3 \cdots\cdots \cap U_n}_{\in \mathcal{B}} \ni \left(\frac{1}{2}, \frac{1}{2}, \ldots\ldots\right)$

but $\Big(\underbrace{\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2}}_{n}, -\frac{1}{10}, -\frac{1}{10} \ldots\Big) \in U$

then $U \not\subset A$

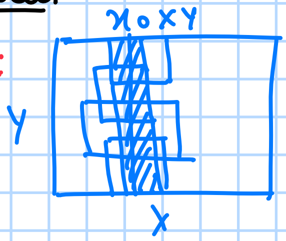
so a typical open set contains a lot of points

thus, In particular if $U_i \neq \mathbb{R}$ and open in $\mathbb{R}$ then

$U_1 \times U_2 \ldots\ldots$ is not open in $\mathbb{R} \times \mathbb{R} \times \cdots\cdots$

21st march:

Recap:



The whole tube gets covered with $U = U_{x_{0_1}} \cap U_{x_{0_2}} \dots \cap U_{x_{0_k}}$

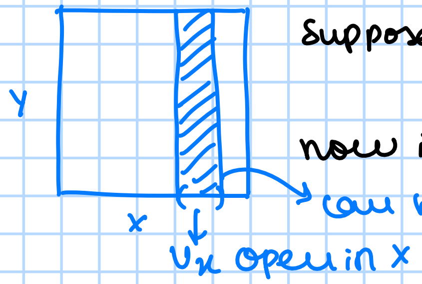
s.t $\underset{\text{open}}{U} \times \underset{\text{open}}{Y}$ is covered (assuming Y is compact with our induction step)

Theorm: If X_α, $\alpha \in I$ is compact $\forall \alpha \in I$, then so is ($\prod_{\alpha \in I} X_\alpha$, product topology)

proof: Firstly we cannot mimic the n = finite proof as for $X \times Y$
we only need $x_0 \times Y$ to be covered
$\Rightarrow$ finitely many rectagles cover it
$\Rightarrow$ And this cover a strip and then we looked at X.

now for $X = X_1$
$Y = X_2 \times X_3 \dots\dots$
$\rightarrow$ we canot garantee Y is compact

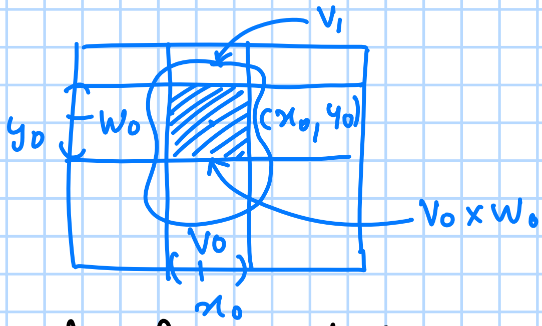
So, we canot just assume Y is compact
this is the issue, so now we will try to prove for $X \times Y$ without using the fact that Y is compact.



suppose $\mathcal{A} = \{V_\alpha \mid \alpha \in I\}$ V_α is open in $X \times Y$ $\forall \alpha \in I$
& $X \times Y = \bigcup_{\alpha \in I} V_\alpha$

now if this has no finite subcover then
then $\exists x_0 \in X$ s.t
no tube $U_{x_0} \times Y = \pi_1^{-1}(U_{x_0})$
i.e $x_0 \in U_{x_0} \rightarrow$ open in X
s.t tube covered by finite subcover

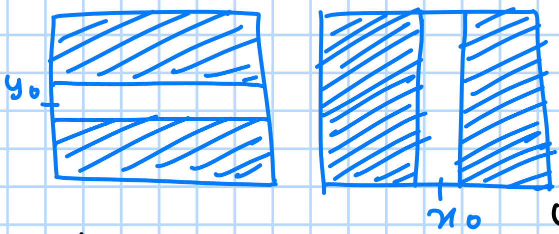
Similarly $\exists y_0 \in Y$ s.t no tube $X \times U_{y_0}$ is covered by finite many open sets from $\{V_\alpha \mid \alpha \in I\}$



now look at $(x_0, y_0) \in X \times Y$
suppose $(x_0, y_0) \in V_1$ for some $V_1 \in \mathcal{A}$

Thus $\exists U_0 \ni x_0$, $W_0 \ni y_0$ both open in X, Y respectively s.t $(x_0, y_0) \in V_0 \times W_0 \subseteq V_1$

now assume also that that The copliment of the tubes $V_0^c \times Y$ and $X \times W_0^c$ are also in $\mathcal{A}$. (grand hypothesis)



Then $\{V_0^c \times Y, X \times W_0^c, V_0 \times W_0\}$ cover $X \times Y$
this is a contradiction.

But the hypothesis grand is too much to expect, so we fix $\mathcal{A} = \{V_\alpha \mid \alpha \in I\}$ be an open cover of $X \times Y$ s.t no finite subcover

Now we enlarge $\mathcal{A}$ by using the following process:
If $B \subseteq X \times Y$ s.t
$B^c \nsubseteq \bigcup_{i=1}^{N} V_\alpha$ for any $N \geq 1$
then we add B to $\mathcal{A}$, and so cosider
$\mathcal{A} \cup \{B\} = \mathcal{A}'$

now $\mathcal{A}'$ is s.t ① $\mathcal{A}'$ will have no finite subcover
② $\mathcal{A}'$ is not necessary a collection of open sets any more

now, $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ be s.t it is the largest collection with property that $X \times Y$ has no finite subcover (Zorns lemma)
($\tilde{A}$ can/cannot have open sets)

and if B^c cannot be coved by finitely many elements $\Rightarrow B \in \tilde{\mathcal{A}}$
(property of $\tilde{\mathcal{A}}$)
(As $\tilde{A}$ is largest $\mathcal{A}' \subseteq \tilde{\mathcal{A}}$)

as $V_0 \times Y$ is s.t it cannot be covd by finite many things
$\Rightarrow V_0^c \times Y \in \tilde{A}$

similarly $X \times W_0^c \in \tilde{A}$

and as $\mathcal{A} \subseteq \tilde{\mathcal{A}} \Rightarrow V_0 \times W_0 \in \tilde{\mathcal{A}}$

So now $\{V_0^c \times Y, X \times W_0^c, V_0 \times W_0\}$ is a finite subcover of $\tilde{\mathcal{A}}$ covering $X \times Y \to$ this is a contradiction.

($A \subseteq \tilde{A} \to$ Zorns lemma
and then as $A \subseteq A' \Rightarrow A' \subseteq \tilde{A} \Rightarrow \tilde{A}$ has property of A')

25th march:

Recall: Defination of compact space

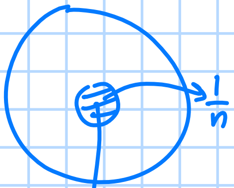
Note: How to come up with compact space examples or non-examples

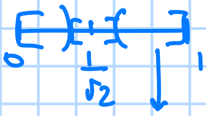
Eg: $[0,1]^d, d \geq 1$ is compact (Tychnoff theorm)

Eg: The non-example is $\mathbb{R}$ for $V = \{(-n,n)\}$ $V_n = (-n,n)$
s.t $\bigcup_{n=1}^{\infty} V_n = \mathbb{R}$ but finite does not

Eg: $(0,1)$ is an example
$V_n = (\frac{1}{n}, 1)$ then $\bigcup_{n=1}^{\infty} (\frac{1}{n}, 1) = (0,1)$
but no finite subcover

Eg: $\mathbb{Q} \cap [0,1]$ let $V_n = \left[\frac{1}{\sqrt{2}} - \frac{1}{n+1}, \frac{1}{\sqrt{2}} + \frac{1}{n+1}\right]^c$





as $\varepsilon \to 0$
this cores all rationals

$V_n = \left[\frac{1}{\sqrt{2}} - \frac{1}{n+1}, \frac{1}{\sqrt{2}} + \frac{1}{n+1}\right]^c$

(general to show A is not compact show $\exists$ limit point of A not in A then we can use this argument)

now if $\exists$ finite subcover
then $\bigcup_{i=1}^{N} V_{n_i} = V_{n_N}$ here let $n_N = n'$
then
$= \left[\frac{1}{\sqrt{2}} - \frac{1}{n'+1}, \frac{1}{\sqrt{2}} + \frac{1}{n'+1}\right]^c$

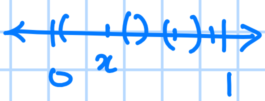
Note: For a limit point of a set, removed then make a set like this and show not compact.

Exe: $A = \{x_1, \ldots\} \subseteq [0,1]$
A is bounded
then if A is not closed then not compact

open cores for A:
$V_n = (x_{i-1}, x_{i+1})$
and $\bigcup_{i=1}^{\infty} (x_i - 1, x_i + 1) \supseteq A$

$\text{diam}(x_i - 1, x_i + 1) = 2$
as $|A| \leq 1$
ofcorse finite subcover

now for $V_i = (x_i - \frac{1}{3}, x_i + \frac{1}{3})$



if V_{x_1} does not cover everyting then $\exists x_2$, $V_{x_1} \cup V_{x_2}$ does not core then $\exists x_3 \ldots x_4$

s.t $V_{x_1} \cup V_{x_2} \cdots \cup V_{x_4}$ cover A
as
$\text{diam}(V_{x_1} \cup V_{x_2} \cup \cdots V_{x_4}) \geq \dim(A)$

now $V_n = \left(x_n - \frac{1}{100(n+1)^2}, x_n + \frac{1}{100(n+1)^2}\right)$ depends on $\{x_1, \ldots x_n\}$

then $\bigcup_{n=1}^{\infty} V_n \supseteq A$
open cover

as if $\{x_1, \ldots x_n\} = \mathbb{Q} \cap [0,1]$ then not but $\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\} \rightarrow$ then possible

Exe: let $A = \mathbb{Q} \cap [0,1]$
take $A = \{x_1, \ldots\}$
& $V_i = \left(x_i - \frac{1}{(2)^{i+2}}, x_i + \frac{1}{(2)^{i+2}}\right) \rightarrow$ open cover

Now we have $A = \mathbb{Q} \cap [0,1]$ s.t

$$V_i = \left(x_i - \frac{1}{(2)^{i+2}}, x_i + \frac{1}{(2)^{i+2}}\right)$$

now as
$\bigcup_{i=1} V_i \supseteq A \rightarrow \{V_i\}$ is an open cover of A

if $A = \{x_1, x_2 \ldots\} \rightarrow$ indexed it s.t $x_1 < x_2 \ldots$

then, $x_n - x_{n-1} > 0$
say $x_n - x_{n-1} = \varepsilon > 0$
then
now if $x_n \in V_{n-1}$ then
$\varepsilon < \frac{1}{(2)^{n+1}}$

if finite subcover exist then let

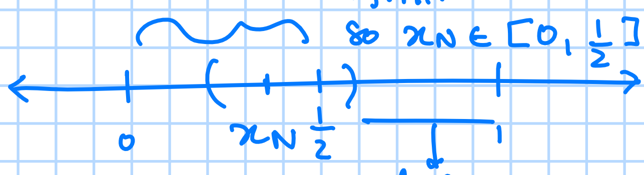
$\{V_{n_1}, \ldots V_{n_r}\}$ be finite subcover

& let $N = \max\{n_1, \ldots n_r\}$ then
as $A \cap [0, \frac{1}{2}]$ contains ∞ numbers
$x_N \in A \cap [0, \frac{1}{2}]$
so $x_N + \frac{1}{2^N+2} < \frac{1}{2} + \frac{1}{2^2} < 1$

so $1 - x_N > \frac{1}{2^{N+2}}$
r
so $1 \notin \bigcup_{i=1}^{r} V_{n_i}$

$\therefore$ Not a finite subcover

$x_1 \ldots\ldots$ infinite
so $x_N \in [0, \frac{1}{2}]$



$\therefore$ No finite subcover

Exe: suppose $A \subseteq [0,1] = \{x_1, \ldots\}$ is not compact, Does there exist $r_n \to 0$ s.t following open cover

$(x_n - r_n, x_n + r_n)$ has no finite subcover now let $\{x_1 \ldots\}$ s.t

then $A = \{x_1, \ldots\}$

x_i is not a limit point

& so $\exists r_i > 0$ s.t

$$B(x_i, r_i) \cap A \setminus \{x_i\} = \phi$$

$\Rightarrow$ this is for all x_i

so $\bigcup_{i=1}^{\infty} B(x_i, r_i) \supseteq A$ and so finite cover

now if $\exists$ finite subcover then as all balls are distinct

$$\{B(x_{n_1}, r_{n_1}), \ldots B(x_{n_r}, r_{n_r})\}$$

we have any x_{n_r+1} not in cover

so, Not possible

One-point compactification:

Suppose X is Hausdorff, can we put it inside a compact space Y, i.e does $\exists\, Y$ s.t

① $X \subseteq Y$ as a subspace

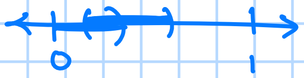
② Y is compact

③ Y is Hausdorff ④ $|Y - X| = 1$ ← only one point

Eg:

① $(0,1)$ is not compact but we can put it inside $[0,1]$ i.e

$$(0,1) \subseteq [0,1]$$



② A not compact but bounded then (A in $\mathbb{R}$)

$$A \subseteq \bar{A}$$

↓

Bounded + closed $\Rightarrow$ compact

③ $\mathbb{R} \subseteq Y$

for $Y = \mathbb{R} \cup \{\infty\}$ as a set, then topology on Y

$$\tau = \underbrace{\{V \mid V \text{ open in } \mathbb{R}\}}_{\text{first condition of } \mathbb{R} \subseteq Y \text{ done}} \cup \{\{\infty\} \cup \mathbb{R} \setminus K \mid K \text{ is compact}\}$$

$\{Y\}$

for any open cover $\exists\, V$ s.t $\{\infty\} \subseteq V$

so $V = K^c$

↓

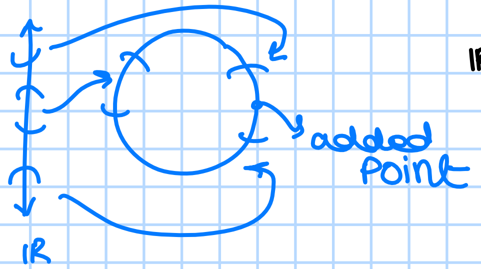
some compact set

but K is cov'd by finitely many

so, finite subcover.

lemma: (Y, τ) satisfies the desired properties (one point compactification of $\mathbb{R}$)

proof:



$\mathbb{R} \cong (1,\infty)$ by $f(t) = 1 + e^{t}$

$f\ (1,\infty) \cong (0,1)$ by $g(s) = \frac{1}{s}$

so $\mathbb{R} \cong (0,1)$

now, $(0,1) \longrightarrow S^1$
$\theta \longmapsto e^{2\pi i \theta}$
is injective, open map

now $\mathbb{R} \longrightarrow S^1$ is s.t

$f(t) = e^{\frac{2\pi i}{1+e^t}}$ is injective open set

$\bigcup_{i \in I} K_i^c = (\bigcap K_i)^c$

$\bigcap_{i=1}^{N} K_i^c = \left(\bigcup_{i=1}^{N} K_i\right)^c$

} to show that Y is a topology

now, to show Y is compact
$\bigcup_{\alpha \in I} V\alpha = Y$ Then $\exists V_{\alpha_0} \ni \infty$

wlog $V_{\alpha_0} = \{\infty\} \cup \mathbb{R} \setminus K$, but as (this is by construction)

K is compact $K \subseteq \bigcup_{i=1}^{N} V_{\alpha_i}$ for some N

$\therefore Y \subseteq \bigcup_{i=0}^{N} V_{\alpha_i}$ → finite subcover

Note: Trivially

$Y = X \cup \{\infty\}$
$\tau = \{V \text{ is open in } X\} \cup \{Y\}$ ← trivially add space itself so if $\infty \in U$ open $\Rightarrow U = Y$

then a cover of Y contains $\{\infty\}$ but that set is Y
so, Y is compact

The problem here is Y is not hausdorff

we want to show this for general X:

lemma: If $\exists Y$ satisfying the condition of first question of one-point c then X must have the following property: $(Y = X \cup \{\infty\})$

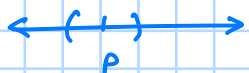
for $p \in X$, $\exists U_p \ni p$ open s.t (open in Y)
$\overline{U_p}$ is compact

locally compact
$(\forall x \in X, \exists U_x \ni x$ s.t
$\overline{U_x}$ is compact$)$

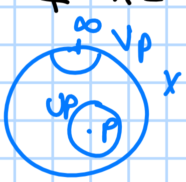
proof: Y is compact & hausdorff

for $X = \mathbb{R}$

& $X \subseteq Y \Rightarrow X$ must also be hausdorff



$U_p = (p-1, p+1)$
$\overline{U_p} = [p-1, p+1]$



as $Y = X \cup \{\infty\}$
for $\infty \in Y$
$p \in Y$
as $\infty \neq p \Rightarrow \exists U_p, V_p$ s.t

$p \in U_p$
$\infty \in V_p$

$U_p \cap V_p = \phi$

then as $U_p \cap V_p = \phi \Rightarrow \infty \notin \overline{U_p}$

so $\overline{U_p}$ is compact (as Y is compact) and

$\overline{U_p} \subseteq X$ as $\infty \notin \overline{U_p}$

Note: This property is called "local compactness"

Defn: (Locally compact) $\forall p \in X, \exists U_p \ni p$ open s.t

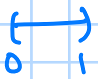
$\overline{U_p}$ is compact

Theorm: If X is locally compact & Hausdorff then

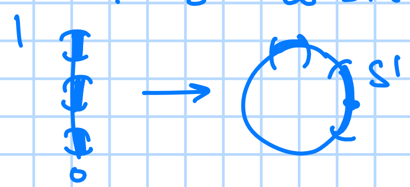
$\exists$ topology on $X \cup \{\infty\} = Y$ so that it

Y is compact, Hausdorff, $X \subseteq Y$ subspace

Eg:

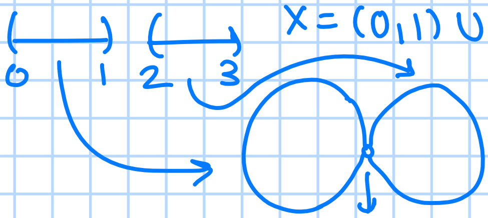


for this
$X = (0,1)$
$Y = S^1$ is s.t



Note: Y is called one-point compactification

Eg:



$X = (0,1) \cup (2,3)$

one point compactification

28^{th} march :

cont fuction :

$[0,1]^2 \longrightarrow \mathbb{R}$

$(x_1, x_2) \longrightarrow x_1$ ← π_1 is cont by def of product space

$(x_1, x_2) \longrightarrow x_2$ ← cont

$\Rightarrow (x_1, x_2) \longrightarrow x_1 + x_2$ ← cont

$$\begin{tikzcd} {[0,1]^{\mathbb{N}}} \arrow[r] \arrow[dr] & \mathbb{R}^2 \ (x_1, x_2) \arrow[d] \\ & \mathbb{R} \ (x_1 + x_2) \end{tikzcd}$$

$f : [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ ← continious

$(x_n) \longrightarrow (x_1 + x_2)$

Note : $\Phi : [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$

s.t $\Phi(x) = \sum_{i=1}^{\infty} x_i$ ← Not even a fuction

does there exist a cont fuction s.t it takes all x_i

If $\Phi(x) = \sup_{n=1}^{\infty} x_i$?

Exe : is $\Phi : [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ above continious?

No, as $\Phi(0,0,\dots) = 0$

if Φ is cont on 0

then $\exists$ open set around $(0,0,\dots)$

this open set looks like:

$$\underbrace{[0,\varepsilon_1) \times [0,\varepsilon_2) \times \cdots}_{\text{after } n} \times [0,1] \times [0,1] \cdots$$

but this means that

$$\Phi(\underbrace{0,0,\dots\dots 0}_{n \text{ times}},1,\dots\dots) = 1$$

$\in$ that open set

for $\varepsilon = 1/2$

$\Phi(U) \subseteq [0,\frac{1}{2})$

but every U contains a point s.t $\Phi = 1$

so, not possible

Now let $(x_n) \overset{\Phi}{\longmapsto} \frac{x_1}{2} + \frac{x_2}{4} + \dots\dots\dots$

now as each x_i is atmost 1

here the tail:

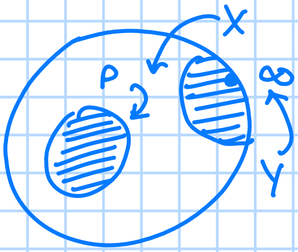
$$\underbrace{\frac{x_N}{2N} + \frac{x_{N+1}}{2N+1} + \dots\dots}_{< \frac{1}{2N+1}}$$

so for N big enough this becomes continious what is $\Phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ cont now

so $\mathbb{R} \to (0,1)$
then
so
$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \cdots & \longrightarrow & \mathbb{R} \\ \downarrow \psi & \nearrow & \\ (0,1) \times (0,1) \times \cdots & & \end{array}$$

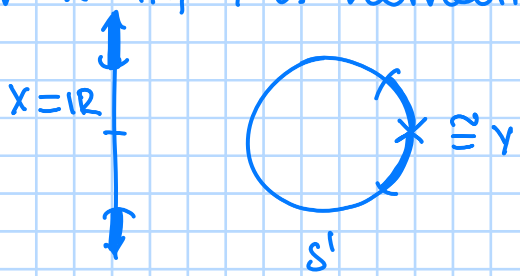
Exe: Now other way, if $f_i : (0,1) \to (0,1)$ is cont, then is it true that $\Phi : (0,1) \to (0,1)^{\mathbb{N}}$ given by
$$\Phi(t) = (f_1(t), \ldots, f_n(t), \ldots) \text{ is continious?}$$
Trivially put $f_i(x) = 0.1$ then $\Phi(t) = (0.1, 0.1, \ldots, 0.1)$ cont as const

Compactification:
so far we did one-point compactification, X is Hausdorff, locally compact



(locally compact: $\forall p \in X$, $\exists U_p \ni p$ open s.t $\overline{U_p}$ is compact)

Eg: for $X = \mathbb{R}$, Y is homeomorphic to S^1



Now we want a new property, $\forall$ cont funtion on X
$\exists F$ cont on Y s.t
$$F|_X = f$$
now we need f to be bounded as otherwise not possible

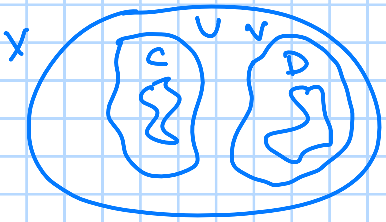
Note: Now for given X we will consider Hausdorff

given X (Hausdorff) can we put it inside Y which is
① compact, Hausdorff
② $\overline{X} = Y$
③ X is a subspace of Y
④ given any continious bounded
$f : X \to \mathbb{R}$
$\exists$ $F : Y \to \mathbb{R}$ cont s.t
$F|_X \equiv f$

(Here we have to drop being greedy, i.e not only ∞ but need more points as we want $F|_X = f$ $\forall f$ cont on \ )

Urysohns Lemma:

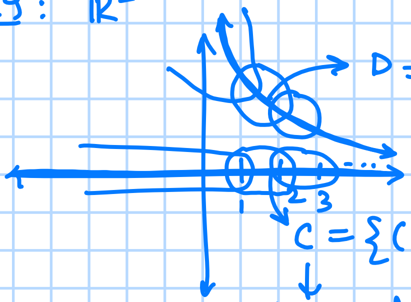
Defn: A space X is called normal if it is hausdorff and given any two disjoint closed sets C, D we seperate them by open sets



$C \subseteq U$
$D \subseteq V$
$U \cap V = \phi$

(normal only if hausdorff)

Eg: $\mathbb{R}^2$



$D = \{ (x, \frac{1}{x}) \mid x \in (0, \infty) \}$

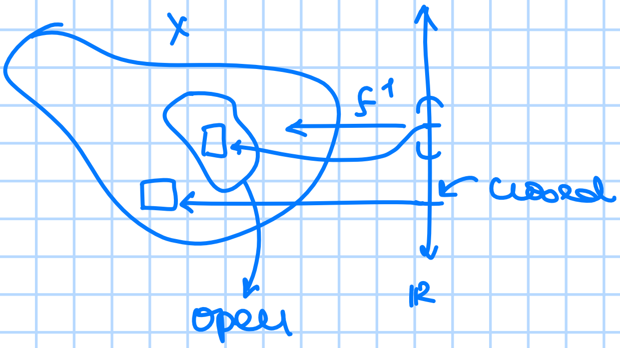
$\updownarrow n \quad h \to 0$

$C = \{ (x, 0) \mid \forall x \in \mathbb{R} \}$

closed set

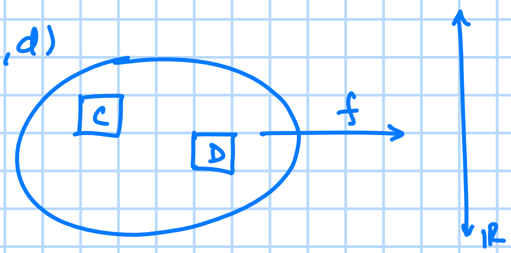
possible in this case also

one way to create open sets over a topology given a cont fuction is that we look at inverse images of that fuction.



Lemma: Every metric space is normal

proof : (X, d)



putting $f(x) = \dfrac{d(x, C)}{d(x, C) + d(x, D)}$

$d(x, A) = \inf_{y \in A} d(x, y)$

(as d is cont on $X \Rightarrow f$ is cont)

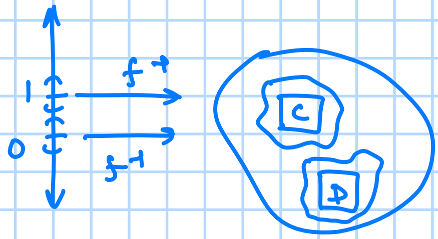
① f is well defined as deominator not zero
② f is cont by triagle inequality

also $\displaystyle f(x)_{x \in C} = 0$

$\displaystyle f(x)_{x \in D} = 1$

now $C = f^{-1}\{0\}$ — cont
$D = f^{-1}\{1\}$ — cont

$\Rightarrow C, D$ are closed



so any $\underbrace{f^{-1}(1-r_1, 1+r_1)}_{\text{open}} \supseteq C$

$\underbrace{f^{-1}(0-r_2, 0+r_2)}_{\text{open}} \supseteq D$

s.t $(1-r_1, 1+r_1) \cap (0-r_2, 0+r_2) = \phi$

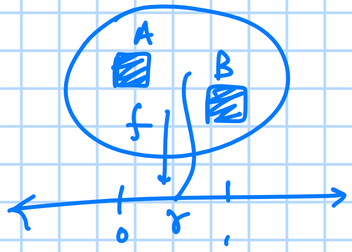
$\Rightarrow f^{-1}(1-r_1, 1+r_1) \cap f^{-1}(0-r_2, 0+r_2) = \phi$
& both open in X

$\therefore$ every metric space is normal

1st Apr :

Theorm : Suppose X is normal, A, B disjoint closed sets in X, then a seperation by open sets can be created by function

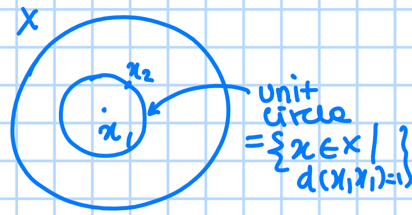
$$\exists f: X \longrightarrow [0,1] \text{ cont}$$
$$f|_A \equiv 0, \quad f|_B \equiv 1$$



proof : let's first prove the special case,

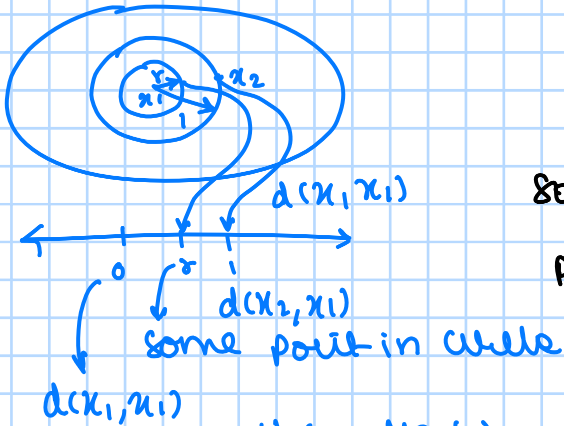
① X is a metric space

② $A = \{x_1\}$ $B = \{x_2\}$ and $d(x_1, x_2) = 1$



now define $f(x) = d(x, x_1)$ is cont by triangle identity

now $f(x_1) = d(x_1, x_1) = 0$

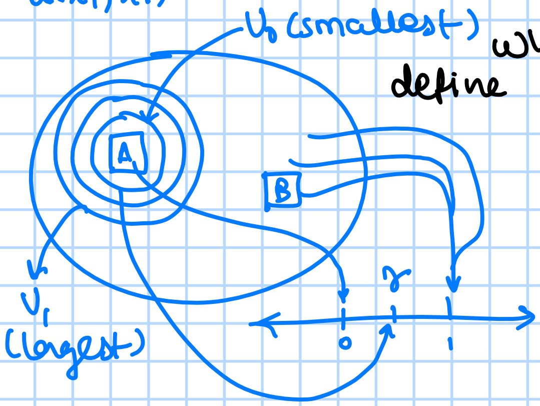


So we take union of spheres and define some f

proof for general case:

assume, $\exists$ collection of open sets U_r, $0 \leq r \leq 1$ s.t $A \subseteq U_0$ $B \subseteq X \setminus U_1$

where $\overline{U_r} \subseteq U_s$ if $r < s$



define

$$f(x) = \begin{cases} \inf\{r \mid x \in U_r\} & ; x \in U_1 \\ 1 & ; x \in X \setminus U_1 \end{cases}$$

now we have to verify the following:

① $f|_A \equiv 0$, $f|_B \equiv 1$, $f(X) \subseteq [0,1]$, f is cont

② $\exists$ such collection of U_r

now, $f|_A \equiv 0$, $f|_B \equiv 1$, $f(X) \subseteq [0,1]$ is trivial, so lets show f is cont

as $f|_{X \setminus U_1} \equiv 1$ $\Rightarrow f$ is cont on $X \setminus U_1$

$\rightarrow$ closed set

now $f|_{\overline{U_1} \setminus U_1} \equiv 1$ so we have to show

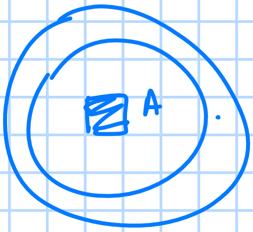
$f|_{\overline{U_1}}$ is cont then we are done

as $\overline{U_1} \cup \overline{U_1} \setminus U_1 \cup X \setminus U_1 = X$

$\downarrow$

f is cont by previous theorm

for some L $x \in \overline{U_1}$, $f(x) = L$ for some $L \in [0,1]$



$$f(U_{r+\varepsilon} \setminus \overline{U_r}) \subseteq [r, r+\varepsilon]$$

then

$0 < L < 1$

$$f(\underbrace{U_{L+\varepsilon/2} \setminus \overline{U}_{L-\varepsilon/2}}_{\text{open}}) \subseteq \left[\ell - \frac{\varepsilon}{2}, \ell + \frac{\varepsilon}{2}\right] \subseteq (\ell-\varepsilon, \ell+\varepsilon) \quad \text{if } \ell < 1$$

$$U_{L+\varepsilon/2} \setminus \overline{U}_{L-\varepsilon/2} \subseteq \underbrace{f^{-1}(L-\varepsilon, L+\varepsilon)}_{\text{open (By definition, not } f^{-1}(U) \text{ open)}}$$

now, $f(X \setminus \overline{U}_{1-\varepsilon})$ for $L = 1$

$$\Rightarrow f(X \setminus \overline{U}_{1-\varepsilon}) \subseteq [1 - \tfrac{\varepsilon}{2}, 1] \subseteq (1-\varepsilon, 1]$$

Similar for $L = 0$

let $\mathbb{Q} \cap [0,1]$, define $\mathbb{Q} \cap [0,1] = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \dots\dots \right\}$

let $U_1 := X \setminus B$ also U_r s.t $r \in \mathbb{Q} \cap [0,1]$

then $B \subseteq X \setminus U_1$

$\exists\, U_A \supseteq A$

$U_B \supseteq B$

$U_A \cap U_B = \phi \Rightarrow$ no limit point of U_A in U_B

so let $U_A = U_0$

then

$$A \subseteq U_0$$

$$\& \ \overline{U_0} \cap B = \phi \Rightarrow \overline{U_0} \cap [X \setminus U_1] = \phi$$

$$\Rightarrow \overline{U_0} \subseteq U_1$$

now create $U_{\frac{1}{2}}$, we want $\overline{U_0} \subseteq U_{1/2}$

$\overline{U_{1/2}} \subseteq U_1$

look at $\overline{U_0}$ = closed &

$\overline{U_0} \cap B = \phi$

$\exists\, U_{1/2} \supseteq \overline{U_0}$ s.t $\overline{U_0} \subseteq U_{1/2}$

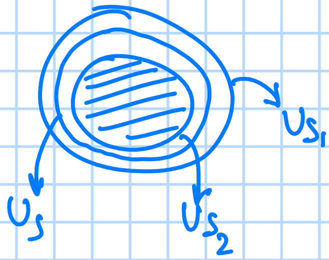
$\overline{U_{1/2}} \cap B = \phi$

$\Rightarrow \overline{U_{1/2}} \subseteq U_1$

n^{th} step: s be nth number of $\mathbb{Q} \cap [0,1]$ then

let $s_1 := \min\{r > s \mid r \text{ has been picked}\}$

$s_2 := \max\{r < s \mid r \text{ has been picked}\}$



we have to show,

$\overline{U_s} \subseteq U_{s_1}$

$\overline{U_{s_2}} \subseteq U_s$

$\overline{U_{s_2}}$ is closed $X \setminus U_{s_1}$ is closed

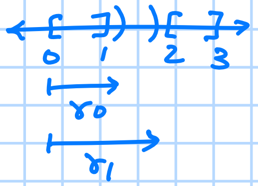
so $\exists\, U_s \supseteq \overline{U_{s_2}}$ s.t

$\overline{U_s} \cap [X \setminus U_{s_1}] = \phi$ By limit point not intersecting

$\Rightarrow \overline{U_s} \subseteq U_{s_1}$

Eg:

$A = [0,1]$
$B = [2,3]$
$\mathbb{R}$



$U_1 = (-1, 1.5) \longrightarrow$ largest
$U_0 = (-0.5, 1.25)$
then $\overline{U_0} \subseteq U_1$
smallest

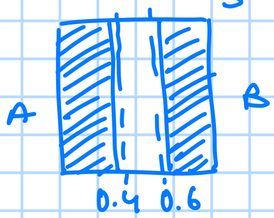
now parametrise:

$$U_r = (-0.5[1-r] - 1[r],\ 1.5(r) + 1.25(1-r))$$

$$U_r = (-0.5 - 0.5r,\ 1.25 + 0.25r)$$

then $U_0 = (-0.5, 1.25)$
$U_1 = (-1, 1.5)$

Eg: $X = [0,1]^2$ $\quad A = [0, \frac{1}{3}] \times [0,1]$ $\quad B = [\frac{2}{3}, 1] \times [0,1]$



$U_0 = [0, 0.4) \times [0,1]$
$U_1 = [0, 0.6) \times [0,1]$

Note: To find examples, parametrise U_r

Eg: $X = [0,1]^{\mathbb{N}}$

$$A = \left\{ x \in X \;\middle|\; \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{n^2} \geq \frac{1}{10} \text{ and } x_n \leq x_{n+1} \right\}$$

we show continuity of this to show A is closed

$$B = \{ x \in X \mid x_n \geq x_{n+1} \}$$

closed as
if $x_n \geq x_{n+1}$
then $f: x_n \to x_{n+1}$
$x_n \geq x_{n+1}$ has a set which is closed
so intersection of all is closed

$$\left\{ f \geq \frac{1}{10} \right\} \subseteq U_0 = \left\{ f > \frac{1}{11} \right\}$$

cont

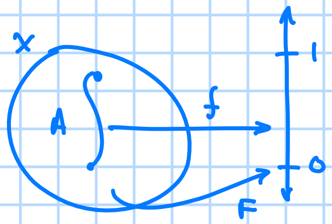
$$U_r = \left\{ f > \frac{1}{11+r} \right\}$$

4th Apr :

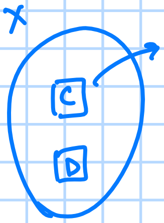
Theorm: (Times extension theorm) X is normal, and A is closed subset of X, $f: A \to [0,1]$ then $\exists$ continious extension F with controled deviation outside A

$F|_A \equiv f$

$F(X) \subseteq [0,1]$ (controlled deviation is stronger condition)



Urson's lemma is just an imediate extension of this



$A = C \sqcup D$

$f|_C \equiv 0$

$f|_D = 1$

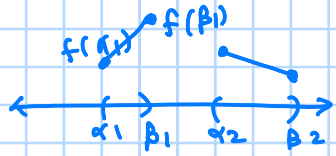
$f: C \sqcup D \longrightarrow [0,1]$

$f|_C \equiv 0, \quad f|_D \equiv 0$

So, $\exists F: X \longrightarrow [0,1]$ s.t $F|_{C \sqcup D} \equiv f$ } Statement of urison's lemma

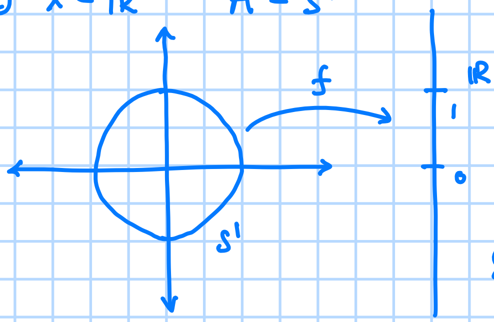
Eg: ① $X = \mathbb{R}$

A is closed in $\mathbb{R}$ $\quad f: A \longrightarrow [0,1]$



$A^c = \bigsqcup_{i \in I} (\alpha_i, \beta_i)$

② $X = \mathbb{R}^2 \quad A = S^1$



$f(x,y) = x^2 + y^2 \equiv 1$ on S'

$F \equiv 1$ works here

if $f(x,y) = x + y$

$f(S') = [-\sqrt{2}, \sqrt{2}]$ ← as connected

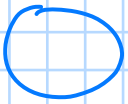
So, $F|_{S'} \equiv f$ and $F(X) = [-\sqrt{2}, \sqrt{2}]$

$F(x,y) = \min(\sqrt{2}, \max(-\sqrt{2}, x+y))$

one more $F(x,y) = \begin{cases} x+y & ; x^2+y^2 \leq 1 \\ \dfrac{x+y}{\sqrt{x^2+y^2}} & ; x^2+y^2 > 1 \end{cases}$

$f(x,y) = x + y$ for $(x,y) \in S'$

now to find F



if f is s.t $\nabla f = 0$ ← Harmonic

then if f is not const and max val of f on boundry

$x + y = \langle (x,y), (1,1) \rangle$

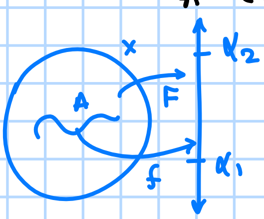
$\leq \|(x,y)\| \, \|(1,1)\| = \sqrt{2}\sqrt{x^2+y^2}$

$\Rightarrow \dfrac{x+y}{\sqrt{x^2+y^2}} \leq \sqrt{2}$

$x+y$ max on bondry and $\dfrac{x+y}{\sqrt{x^2+y^2}} \leq \sqrt{2}$ outside circle

proof: special cases:

$A = C \sqcup D$, C, D closed then $f|_C \equiv 0$, $f|_D \equiv 1$, use urson lemma to get F



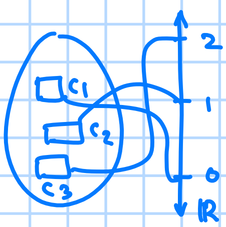
$A = C \sqcup D$

when $f|_C \equiv \alpha_1$

$f|_D \equiv \alpha_2$

apply urson's lemma to get $F_1(X) \subseteq [0, \alpha_2 - \alpha_1]$

then $F(x) = \alpha_1 + F_1(x)$



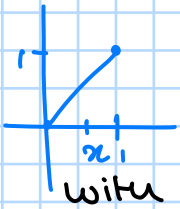
Find $F_2: X \longrightarrow \mathbb{R}$ so that $F_2|_{C_2} \equiv 1 \quad F_2|_{C_1 \sqcup C_3} \equiv 0$

Find $F_3: X \longrightarrow \mathbb{R}$, $F_3|_{C_3} \equiv 2$, $F_3|_{C_1 \sqcup C_2} \equiv 0$

now $F = F_2 + F_3$ ← Range of $F \subseteq [0,2]$ By squeezing support of F_2 and F_3

we cannot do what we did with special case as:

$X = \mathbb{R}$
$A = [0,1]$
$f(x) = x$



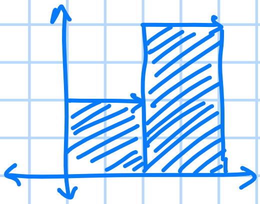
we will fix this hole with integration in the next class

Rieman Integration and Lebsegue integration:

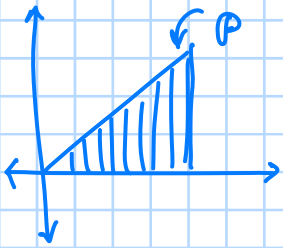
Riemann $f : [0,1] \longrightarrow \mathbb{R}$ continious

if $f \equiv 1$
$\int_0^1 f = 1$

If $f = \begin{cases} 1 & \text{on } [0, 1/2) \\ 2 & \text{on } [1/2, 1] \end{cases}$



now if $f(x) = x$



$1/n$ term partition

$\int_0^1 f = \lim_{n \to \infty} L(P_n, f)$

if $f = \begin{cases} 1 & ; \ \mathbb{Q} \cap [0,1] \\ -1 & ; \ \mathbb{Q}^c \cap [0,1] \end{cases}$

then $U(f, P) = 1 \ \forall P \in \mathcal{P}$
$L(f, P) = -1 \ \forall P \in \mathcal{P}$

so we don't have rieman integral of f

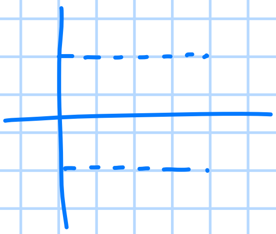
This is where lebegue integration comes where we partition over the range

$\text{Range}(f) = 0 < \alpha_1 < \alpha_2 \cdots < \alpha_n = 1$

then

$S_n(f) = \alpha_1 |f^{-1}(0, \alpha_1)|$
$+ \alpha_2 |f^{-1}(\alpha_1, \alpha_2)|$
$+ \vdots$
$\alpha_n |f^{-1}(\alpha_{n-1}, \alpha_n)|$

so on $f = \begin{cases} 1 & ; \mathbb{Q} \cap [0,1] \\ -1 & ; \mathbb{Q}^c \cap [0,1] \end{cases} = \chi_{\mathbb{Q}} - \chi_{\mathbb{Q}^c}$

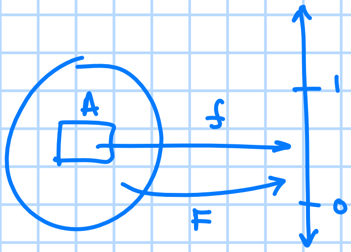


$\text{Range}(f) = \{-1, 1\}$

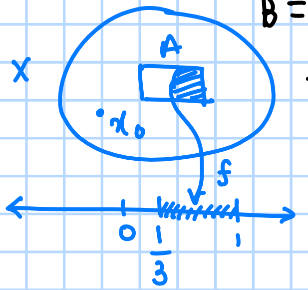
8th April:

Theorm: (Tietze extension theorm) X is normal, and A is closed subset of X, $f: A \to [0,1]$ then $\exists$ continious extension F with controlled deviation outside A

$$F|_A \equiv f$$
$$F(X) \subseteq [0,1]$$



proof: Step 1:



$B = f^{-1}[\frac{1}{3}, 1] \cap A$ is closed

$= f^{-1}[\frac{1}{3}, 1]$

now $\{x_0\}$ and $f^{-1}([\frac{1}{3}, 1])$ are closed and disjoint, so by urson's lemma

$\exists$ F_1 s.t

$$F_1|_{\{x_0\}} = 0$$
$$F_1|_{f^{-1}[\frac{1}{3}, 1]} = \frac{1}{3}$$

now, $|F_1| \leq \frac{1}{3}$, F_1 is continious

and $|f - F_1| \leq \frac{2}{3}$ by construction

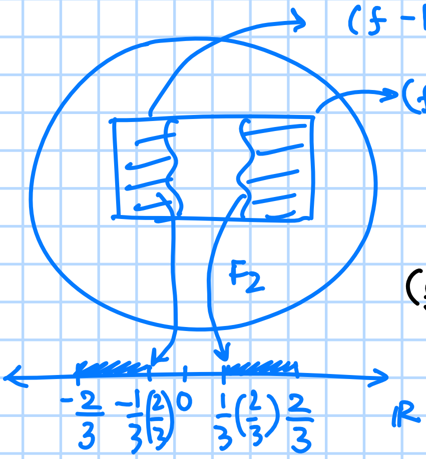
as $f(A) \subseteq [0,1]$

$F_1(A) \subseteq [0, \frac{1}{3}]$

so $|f - F_1| \leq \frac{2}{3}$ on A

$(f - F_1)^{-1}[-\frac{2}{3}, -\frac{1}{3}(\frac{2}{3})]$

$(f - F_1)^{-1}[\frac{1}{3}(\frac{2}{3}), \frac{2}{3}]$



$(f - F_1) \subseteq [-\frac{2}{3}, \frac{2}{3}]$

so, $(f - F_1)^{-1}[\frac{1}{3}(\frac{2}{3}), \frac{2}{3}]$

& $(f - F_1)^{-1}[-\frac{2}{3}, -\frac{1}{3}(\frac{2}{3})]$

now $|F_2| \leq \frac{1}{3}(\frac{2}{3})$

$|f - F_1 - F_2| \leq (\frac{2}{3})^2$

as $F_2 \subseteq [-\frac{1}{3}(\frac{2}{3}), \frac{1}{3}(\frac{2}{3})]$ (By ursons lemma)

and $f - F_1 \subseteq [-\frac{2}{3}, \frac{2}{3}]$

$\Rightarrow f - F_1 - F_2 \subseteq [-\frac{2}{3}, -\frac{1}{3}(\frac{2}{3})] \cup [\frac{1}{3}(\frac{2}{3}), \frac{2}{3}]$

$\Rightarrow |f - F_1 - F_2| \leq (2/3)^2$

So now, $|f - F_1 - F_2| \leq \left(\frac{2}{3}\right)^2$

do this n times to get

$F_n : X \longrightarrow \mathbb{R}$

we get $|F_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

and $|f - \sum_{i=1}^{n} F_i| \leq \left(\frac{2}{3}\right)^n$

now define $G_n = F_1 + F_2 + \cdots F_n : X \longrightarrow \mathbb{R}$

where $G_n \longrightarrow F : X \longrightarrow [-1,1]$ $(\because |F| \leq 1)$

and $G_n \to f$ on A

thus $F|_A = f$

and it coverges as $|F_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

$$\lim_{n\to\infty} |\Sigma F_n| \leq \lim_{n\to\infty} \Sigma |F_n| \leq \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right]$$

$$= \frac{1}{3}\left(\frac{1}{\frac{1}{3}}\right) = 1$$

$$\Rightarrow |F| \leq 1$$

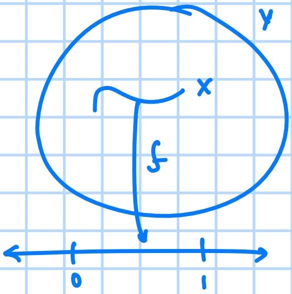
So now insted of $f \subseteq [0,1]$

take $f' = 2f - 1 \subseteq [-1,1]$

then if we scale back

we get $F \subseteq [0,1]$

Given X, does $\exists Y$ s.t Y is compact and:



If X is normal

then $\forall f : X \to [0,1]$ continious

we can do this

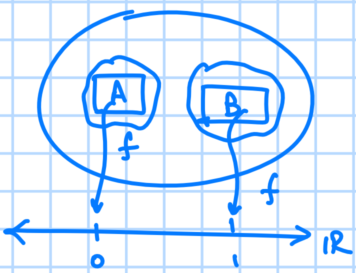
i.e find F

we demand : ① Y is compact and Hausdorff

② $\overline{X} = Y$

③ $\forall f : X \longrightarrow [0,1]$, $\exists F : Y \longrightarrow [0,1]$ s.t
(cont) $F|_X = f$ (cont F)

Note : our demand is yes, given that X is normal



A special case is $X=(0,1)$, then lets find Y s.t $X \subseteq Y$, Y is compact Hausdorff, $\overline{(0,1)} = Y$ and $\forall f$ cont, $\exists F$

where trivially $\overline{(0,1)} = [0,1] = Y$ cannot be taken

as $f(t) = \sin\left(\frac{2\pi}{t}\right)$

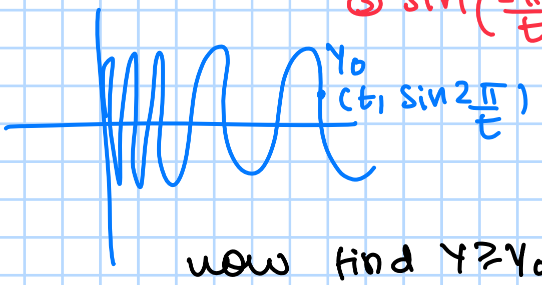
no F exist

lets find $Y \supseteq (0,1)$ s.t

① Y is compact and Hausdorff

② $\overline{(0,1)} = Y$

③ $\sin\left(\frac{2\pi}{t}\right)$ has a cont extension for Y



now find $Y \supseteq Y_0$ s.t $(x,y) \mapsto y$ is cont extension and Y is compact Hausdorff

$\Phi : (0,1) \longrightarrow \mathbb{R}^2$

$\Phi(t) = \left(t, \sin\frac{2\pi}{t}\right), \quad 0<t<1$

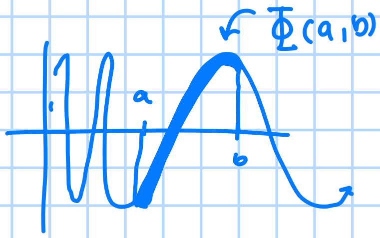
now Φ is continious, Injective, open map for $(0,1) \longrightarrow \Phi(0,1)$

continious: As comp are cont

Injective: yes (trivial)

open map: we have to prove this

$\Phi(a,b)$ is open in $\Phi(X) \subseteq \mathbb{R}^2$



$$\Phi(a,b) = \left\{ \left(t, \sin\frac{2\pi}{t}\right) \middle| a<t<b \right\}$$
$$= \underbrace{\pi_1^{-1}(a,b) \cap \Phi(X)}_{\text{open in product topology}}$$

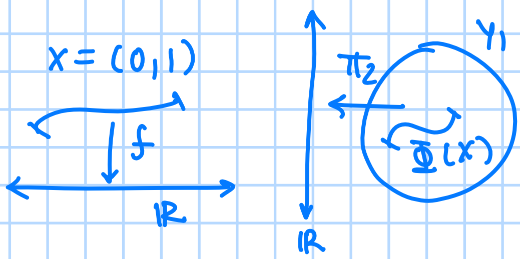
now, $Y_1 = \overline{\Phi(0,1)}$ is compact

$Y_0 = \Phi(0,1)$

$\pi_2 : Y_0 \longrightarrow \mathbb{R}$

$\pi_2(x,y) = y$

$\pi_2\big|_{Y_0} \equiv \sin\frac{2\pi}{t}$



$\pi_2\big|_{\Phi(X)} \equiv f$

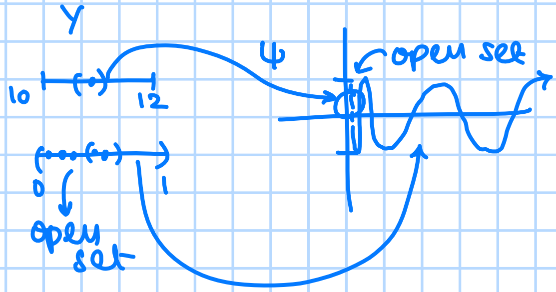
now, let Y: $(0,1) \cup [10,12]$

where

$\Theta : [10,12] \longrightarrow \overline{\Phi(X)} \setminus \Phi(X)$ is bijective define: $\psi : (0,1) \cup [10,12] \longrightarrow Y_1$

$$\psi(t) = \begin{cases} (t, \sin\frac{2\pi}{t}) ; & 0<t<1 \\ \theta(t) ; & t \in [10,12] \end{cases}$$

now make $Y = (0,1) \cup [10,12]$ a topological space by declaring $\psi : Y \longrightarrow Y_1$ a Homeomorphism



$t \mapsto (t, \sin\frac{2\pi}{t})$

now
$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[dr, "F"'] & Y_1 \arrow[d, "\pi_2(x,y)"] \\ & \mathbb{R} \end{tikzcd}$$

F is cont as ψ is homomorphic and π_2 is cont

$$F|_{(0,1)} = \pi_2(t, \sin\tfrac{2\pi}{t}) = \sin\tfrac{2\pi}{t} \equiv f$$

so, $Y \xrightarrow{\psi} Y_1$ Y is topology generated by making ψ hom

Note: If two funtions are given, we go to $\mathbb{R}^3$ and see its graph

$\Phi(X) = Y_0$

$Y_1 \subseteq \mathbb{R}^3$ s.t $Y_1 = \overline{\Phi(X)}$ (compact, Hausdorff)

For general case: (X is normal)

$\mathcal{A} = \{ f : X \to [0,1] \mid f \text{ is cont} \}$

$J = |\mathcal{A}|$

then define $\Phi : X \to [0,1]^J$ s.t

$$\Phi(t) = (f_\alpha(t))_{\alpha \in \mathcal{A}}$$

Φ is injective (we use X is normal here)

Φ is continious (trivial)

Suppose Φ is open-map from X to $\Phi(X) \subseteq [0,1]^J$

then $Y_1 = \overline{\Phi(X)} \subseteq \underbrace{[0,1]^J}_{\text{compact by Ticknoff}}$

closure $\Rightarrow \overline{\Phi(X)}$ is compact

so, Y_1 is compact, Hausdorff

now define A disjoint from X
and
$\theta : A \longrightarrow \overline{\Phi(X)} \setminus \Phi(X)$

define map $\psi : X \cup A = Y \longrightarrow Y_1$

$$\psi(x) = \begin{cases} \Phi(x) ; & x \in X \\ \theta(x), & x \in A \end{cases}$$

and coinduced topology on Y s.t

$Y \xrightarrow{\psi} Y_1$ ψ becomes homeomorphic

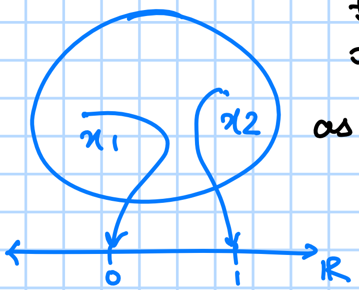
now,
$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[dr, "F_{\alpha_0}"'] & \overline{\Phi(X)} = Y_1 \arrow[d, "\pi_{\alpha_0}"] \\ & {[0,1]} \end{tikzcd}$$
$(x\alpha) \to x\alpha_0$

for any α_0, $f_{\alpha_0} : X \longrightarrow [0,1]$

$$F_{\alpha_0}|_X = \pi_{\alpha_0}(\psi(x))$$
$$= \pi_{\alpha_0}(f_\alpha(x))$$
$$= f_{\alpha_0}(x)$$

$\Phi : X \longrightarrow [0,1]^J$ is injective as:
$x_1 \neq x_2 \in X$
then $\Phi(x_1) \neq \Phi(x_2) \in [0,1]^J$

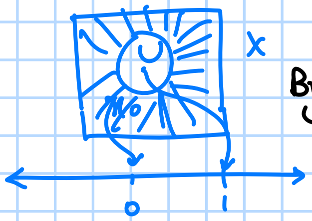
$\Leftrightarrow$ $\exists f_\alpha : X \to [0,1]$ s.t
$f(x_1) = 0$
$f(x_2) = 1$



as X is normal, this is true

for openness :
suppose U is open in X
then $\Phi(U)$ should be open in $\Phi(X) \subseteq [0,1]^J$

we want W open in $[0,1]^J$ s.t $\underbrace{W \cap \Phi(X)}_{\text{open}} \subseteq \Phi(U)$



By normality $\{x_0\}$, $X \setminus U$ are closed
so, $\exists$ f_α cont s.t

$f_\alpha|_{X \setminus U} \equiv 0$ $f_\alpha|_{\{x_0\}} \equiv 1$

$W = \pi_{\alpha_0}^{-1}(0,1] = \{(x_\alpha) \mid x_{\alpha_0} > 0\}$

now $W \cap \Phi(X) \subseteq \Phi(U)$

Endsem tips: Start with Hausdorff, delete points, the above Y is called Stone Weck compactification

what does Y look like

and $X, f_1 \dots f_{1000}$

so does there exist point weak s.t $[0,1]^{10}$

think of what happens to $\dim(J)$ when points are deleted or added,

Homeomorphism & Quotient maps (wk)

Quotient map : $S^2 \to S^1$ (think of examples)

$\mathbb{R} \to \mathbb{R}^2$

torus $\to$ something else